Quiver Presentations for Descent Algebras of Exceptional Type

Götz Pfeiffer
Department of Mathematics
National University of Ireland, Galway
Ireland
goetz.pfeiffer@nuigalway.ie

Mathematics Subject Classification: 20F55; 20F05; 20C40

Abstract

The descent algebra of a finite Coxeter group W is a basic algebra, and as such it has a presentation as quiver with relations. In recent work, we have developed a combinatorial framework which allows us to systematically compute such a quiver presentation for a Coxeter group of a given type. In this article, we use that framework to determine quiver presentations for the descent algebras of the Coxeter groups of exceptional or non-crystallographic type, i.e., of type E_6, E_7, E_8, F_4, H_3, H_4 or $\mathsf{I}_2(m)$.

1 Introduction.

Let W be a finite Coxeter group. The descent algebra $\Sigma(W)$ is a subalgebra of the group algebra of W, arising from the partition of W into descent classes [13]. It supports an algebra homomorphism with nilpotent kernel into the (commutative) character ring of W and therefore is a basic algebra. Each basic algebra has a presentation as quiver with relations [1, Section II.3]. In recent work [8] we have presented an algorithm for the construction of such a quiver presentation for any finite Coxeter group. The algorithm has been implemented in the GAP [12] package ZigZag [9], which is based on the CHEVIE [6] package for finite Coxeter groups and Iwahori–Hecke algebras.

In this article, the algorithm is applied to the Coxeter groups of exceptional or non-crystallographic type. The results for dihedral groups, i.e., Coxeter groups of type $\mathsf{I}_2(m)$, are shown in Section 2, the results for type H_3 in Section 3, for type H_4 in Section 4, for type F_4 in Section 5, for type E_6 in Section 6, for type E_7 in Section 7, and for type E_8 in Section 6. Each such section contains a labelling of the generators of W in the form of a Coxeter diagram, tabular listings of the vertices and edges of the quiver, a picture of the corresponding graph, a list of relations (if any), a listing of the projective indecomposable modules and their Loewy series (verifying results on the Loewy length by Bonnafé and the author [4]), as well as the Cartan matrix.

In this section we briefly recall the setup [8] and provide some important general results. For a general introduction to Coxeter groups and related topics we refer to the books [5] and [7].

1.1 The descent algebra and some of its bases.

Let W be a finite Coxeter group, generated by a set S of simple reflections with corresponding length function ℓ. The descent algebra $\Sigma(W)$ is defined as a subspace of the group algebra $\mathbb{Q}W$ as follows.

For a subset $J \subseteq S$ we denote by W_J the (parabolic) subgroup of W generated by J and we define

$$X_J := \{x \in W : \ell(sx) > \ell(x) \text{ for all } s \in J\}$$

as the set of minimal length right coset representatives of W_J in W. Then

$$X_J^{-1} := \{x^{-1} : x \in X_J\}$$

is a set of left coset representatives of W_J in W and it is well-known that for $J, K \subseteq S$ the intersection

$$X_{JK} := X_J \cap X_K^{-1}$$

is a set of representatives of the W_J, W_K-double cosets in W. For $J \subseteq S$, we form the sum

$$x_J := \sum X_J^{-1} \in \mathbb{Q}W$$

and define $\Sigma(W)$ to be the subspace of $\mathbb{Q}W$ spanned by the elements x_J, $J \subseteq S$. By Solomon's Theorem [13], $\Sigma(W)$ is in fact a subalgebra of $\mathbb{Q}W$, since, for $J, K \subseteq S$,

$$x_J x_K = \sum_{L \subseteq S} a_{JKL} x_L$$

where $a_{JKL} = |X_{JKL}|$ and $X_{JKL} = \{d \in X_{JK} : J^d \cap K = L\}$.

The linear independence of the x_J is best seen with a different basis. For this, let

$$\mathcal{D}(w) := \{s \in S : \ell(sw) < \ell(w)\}$$

be the (left) descent set of $w \in W$. Then $X_J = \{w \in W : \mathcal{D}(w) \cap J = \varnothing\}$. We further define, for $K \subseteq S$, the descent class

$$Y_K := \{w \in W : \mathcal{D}(w) = S \setminus K\} = \mathcal{D}^{-1}(S \setminus K),$$

and we form the sum

$$y_K := \sum Y_K^{-1} \in \mathbb{Q}W.$$

Then $X_J = \coprod_{K \supseteq J} Y_K$ and thus $x_J = \sum_{K \supseteq J} y_K$. Hence, by Möbius inversion,

$$y_K = \sum_{J \supseteq K} (-1)^{|J-K|} x_J$$

and so the y_K span all of $\Sigma(W)$. Clearly the y_K are linearly independent and thus $\Sigma(W)$ has dimension $2^{|S|}$, the number of subsets of S.

The group W acts on itself and on its subsets by conjugation and thus partitions the power set $\mathcal{P}(S)$ into classes of conjugate subsets of S. In this context, a third basis $\{e_L : L \subseteq S\}$ of $\Sigma(W)$ has been introduced by Bergeron, Bergeron, Howlett and Taylor [2], as follows. For $J \subseteq S$, we define

$$X_J^\sharp := \{x \in X_J : J^x \subseteq S\}$$

and we denote by

$$[J] := \{J^x : x \in X_J^\sharp\} \subseteq \mathcal{P}(S)$$

the class of J. Moreover, we denote by

$$\Lambda := \{[J] : J \subseteq S\}$$

the set of all classes, and for $L \in \lambda \in \Lambda$ we set $\|\lambda\| := |L|$.

For $J, K \subseteq S$, we set

$$m_{KL} := \sum_{J \in [K]} a_{JKL} = \begin{cases} |X_K \cap X_L^\sharp|, & \text{if } K \supseteq L, \\ 0, & \text{otherwise.} \end{cases} \tag{$*$}$$

Then, for a suitable ordering of the subsets of S, the matrix $(m_{KL})_{K,L \subseteq S}$ is a triangular matrix with nonzero diagonal entries $|X_J^\sharp| > 0$ and hence invertible. Therefore, the conditions

$$x_K = \sum_L m_{KL} e_L,$$

uniquely define a basis $\{e_L : L \subseteq S\}$ of $\Sigma(W)$.

Among many other properties, Bergeron, Bergeron, Howlett and Taylor [2] show that the elements

$$e_\lambda = \sum_{L \in \lambda} e_L, \quad \lambda \in \Lambda,$$

form a complete set of primitive, pairwise orthogonal idempotents of $\Sigma(W)$.

1.2 The central case.

The finite group W has a unique element w_0 of maximal length which is characterized by the property that $\mathcal{D}(w_0) = S$. We now show how the structure of $\Sigma(W)$ is constrained if w_0 is central in W. With the exception of E_6 and $I_2(m)$, m odd, all examples of Coxeter groups considered here have a central longest element w_0. The results on the Loewy length of $\Sigma(W)$ [4] have shown already that it makes a big difference, whether w_0 is central or not. Here we show that in these cases the descent algebra $\Sigma(W)$ is a direct sum of subalgebras $\Sigma(W)^+$ and $\Sigma(W)^-$, spanned by the idempotents e_λ with $\|\lambda\|$ even, and the e_λ with $\|\lambda\|$ odd, respectively. For W of type B_n, this decomposition of $\Sigma(W)$ has been observed by Bergeron [3].

For later use, we list here some properties of the intersection of $X_L^\sharp$ and Y_K. For $J \subseteq S$, we denote by w_J the longest element of the parabolic subgroup W_J of W. We call $u \in W$ a prefix of $w \in W$, and write $u \leqslant w$, if $\ell(u^{-1}w) = \ell(w) - \ell(u)$. For $J \subseteq S$, the set X_J can then be described as the set of all prefixes of the longest coset representative $w_J w_0$, and Y_J is the set of all $x \in X_J$ which have $w_{S \setminus J}$ as prefix. In other words, Y_J is the interval from $w_{S\setminus J}$ to $w_J w_0$ in the weak Bruhat order.

Lemma 1 *Let* $L \subseteq K \subseteq S$.

(i) $(-1)^{|K|}\,|Y_K \cap X_L^\sharp| = \sum_{J \supseteq K} (-1)^{|J|}\,|X_J \cap X_L^\sharp|$.

(ii) *If* $x \in Y_K \cap X_L^\sharp$ *then* $w_L w_{L\cup(S\setminus K)}$ *is a prefix of* x.

(iii) *In particular,* $Y_L \cap X_L^\sharp = \{w_L w_0\}$.

Proof. (i) From $X_J = \bigsqcup_{K \supseteq J} Y_K$, it follows that

$$X_J \cap X_L^\sharp = \bigsqcup_{K \supseteq J} Y_K \cap X_L^\sharp,$$

and hence by Möbius inversion that $|Y_K \cap X_L^\sharp| = \sum_{J\supseteq K} (-1)^{|J|-|K|}\,|X_J \cap X_L^\sharp|$.

(ii) Let $x \in Y_K \cap X_L^\sharp$. Then $\mathcal{D}(x) = S \setminus K$. We show that if $x \in X_L^\sharp$ then $\mathcal{D}(x) \subseteq \mathcal{D}(ux)$ for all $u \in W_L$. It then follows, for $u = w_L$, that $L \cup \mathcal{D}(x) \subseteq \mathcal{D}(w_L x)$. For any $w \in W$, it is known [7, Lemma 1.5.2] that if $J \subseteq \mathcal{D}(w)$ then $w_J \leqslant w$. Hence $w_{L\cup(S\setminus K)} \leqslant w_L x$ and, since w_L is a common prefix of both sides, $w_L w_{L\cup(S\setminus K)} \leqslant x$.

In order to show that $\mathcal{D}(x) \subseteq \mathcal{D}(ux)$, we argue as follows. For $x \in X_L^\sharp$, it is known [7, Theorem 2.3.3] that if $s \in \mathcal{D}(x)$ then $d := w_L w_{L\cup\{s\}}$ is a prefix of x. Now $d \in X_L^\sharp$ and thus $\ell(u^d) = \ell(u)$. It follows that $\ell(d^{-1}ux) = \ell(u^d d^{-1}x) \leqslant \ell(u) + \ell(x) - \ell(d) = \ell(ux) - \ell(d)$. On the other hand, $\ell(ux) - \ell(d) \leqslant \ell(d^{-1}ux)$. Hence d is a prefix of ux. By construction, $s \in \mathcal{D}(d)$. Thus $s \in \mathcal{D}(ux)$.

(iii) If $L = K$ then $w_{L\cup(S\setminus K)} = w_0$. And $x \in X_L$ implies $x \leqslant w_L w_0$, whereas from (ii) it follows that $w_L w_0 \leqslant x$ for all $x \in Y_L \cap X_L^\sharp$. □

We can now describe the structure of the descent algebra $\Sigma(W)$ for a Coxeter group W with central longest element w_0.

Theorem 1 *Suppose W is a finite Coxeter group with central longest element w_0. If $\lambda, \mu \in \Lambda(W)$ are such that $\|\lambda\| \not\equiv \|\mu\| \pmod 2$ then $e_\mu \Sigma(W) e_\lambda = 0$.*

Proof. The longest element w_0 is the unique element with descent set S and therefore, $w_0 = y_\varnothing \in \Sigma(W)$. Hence, using $(*)$, Lemma 1 (i) and (iii),

$$\begin{aligned} w_0 &= \sum_{K \subseteq S} (-1)^{|K|} x_K \\ &= \sum_K (-1)^{|K|} \sum_L m_{KL} e_L \\ &= \sum_L \sum_{K \supseteq L} (-1)^{|K|} \, |X_K \cap X_L^\sharp| \, e_L \\ &= \sum_L (-1)^{|L|} \, |Y_L \cap X_L^\sharp| \, e_L \\ &= \sum_L (-1)^{|L|} e_L \\ &= \sum_{\lambda \in \Lambda} (-1)^{\|\lambda\|} e_\lambda . \end{aligned}$$

By our hypothesis w_0 is central in W, and so it follows that

$$w_0 e_\lambda = e_\lambda w_0 = (-1)^{\|\lambda\|} e_\lambda$$

for all $\lambda \in \Lambda$. Hence, if $a \in e_\mu \Sigma(W) e_\lambda$ and $(-1)^{\|\lambda\| + \|\mu\|} = -1$ then

$$a = e_\mu a e_\lambda = e_\mu w_0 a w_0 e_\lambda = (-1)^{\|\mu\|} e_\mu a (-1)^{\|\lambda\|} e_\lambda = -e_\mu a e_\lambda = -a,$$

and thus $a = 0$. □

As a consequence, the algebra $\Sigma(W)$ is the direct sum of subalgebras $\Sigma(W)^+$ and $\Sigma(W)^-$, generated by the orthogonal central idempotents

$$\epsilon^+ = \tfrac{1}{2}(\mathrm{id} + w_0) = \sum_{\|\lambda\| \text{ even}} e_\lambda$$

and

$$\epsilon^- = \tfrac{1}{2}(\mathrm{id} - w_0) = \sum_{\|\lambda\| \text{ odd}} e_\lambda ,$$

respectively. We refer to $\Sigma(W)^+$ and $\Sigma(W)^-$ as the even and the odd part of $\Sigma(W)$. In this article, where appropriate, we present results about $\Sigma(W)$ more compactly in terms of the even part $\Sigma(W)^+$ and the odd part $\Sigma(W)^-$. For example, the quiver of $\Sigma(W)$ is described as union of the quivers of $\Sigma(W)^+$ and $\Sigma(W)^-$. And the Cartan matrix of $\Sigma(W)$ is described in the form of two smaller Cartan matrices, thus omitting entries which are 0 due to Theorem 1.

1.3 Quiver presentations.

The descent algebra $\Sigma(W)$ supports an algebra homomorphism θ into the character ring $R(W)$, defined for $J \subseteq S$ by

$$\theta(x_J) = 1_{W_J}^{W}, \tag{1}$$

the permutation character of the action of W on the cosets of the parabolic subgroup W_J. Due to Solomon [13], the kernel of θ coincides with the Jacobsen radical of $\Sigma(W)$. It follows that all simple modules of $\Sigma(W)$ are 1-dimensional and thus that $\Sigma(W)$ is a basic algebra. As such it has a presentation as path algebra of a quiver with relations. Here, we present such quiver presentations for the finite irreducible Coxeter groups of exceptional or non-crystallographic type.

For a general introduction to quivers in the representation theory of finite dimensional algebras we refer to the book [1]. Here, a quiver is a directed multigraph $Q = (V, E)$, consisting of a vertex set V and an edges set E, together with two maps $\iota, \tau \colon E \to V$, assigning to each edge $e \in E$ a source $\iota(e) \in V$ and a target $\tau(e) \in V$. A path of length $\ell(a) = l$ in Q is a pair

$$a = (v; e_1, e_2, \dots, e_l) \tag{2}$$

consisting of a source $v \in V$ and a sequence of l edges $e_1, e_2, \dots, e_l \in E$ such that $\iota(e_1) = v$ and $\iota(e_i) = \tau(e_{i-1})$ for $i = 2, \dots, l$. Let $\mathcal{A}$ be the set of all paths in Q. The vertices $v \in V$ can be identified with the paths $(v; \varnothing)$ of length 0, and the edges $e \in E$ can be identified with the paths $(\iota(e); e)$ of length 1. Concatenation of paths defines a partial multiplication on $\mathcal{A}$ as

$$(v; e_1, \dots, e_l) \circ (v'; e'_1, \dots, e'_{l'}) = (v; e_1, \dots, e_l, e'_1, \dots, e'_{l'}),$$

provided that $\tau(e_l) = v'$.

The path algebra A of the quiver Q is defined as

$$A = \mathbb{Q}[\mathcal{A}], \tag{3}$$

where $a \circ a' = 0$ if the product $a \circ a'$ is not defined in $\mathcal{A}$, and otherwise multiplication is extended by linearity from $\mathcal{A}$.

In recent work [8], the descent algebra is constructed as a subquotient of the quiver algebra defined by the Hasse diagram of the power set $\mathcal{P}(S)$ with respect to reverse inclusion as follows. For $L \subseteq S$ and $s \in S$, denote

$$L_s = L \setminus \{s\}. \tag{4}$$

A path in the quiver is a sequence $L \to L_s \to (L_s)_t \to \dots$, for some subset $L \subseteq S$ and elements $s, t, \dots \in L$, which we write as a pair $(L; s, t, \dots)$. The length of the path $(L; s, t, \dots)$ then is $|\{s, t, \dots\}|$. For each such path and each $r \in S$, we define

$$(L; s, t, \dots).r = (L^d; s^d, t^d, \dots), \tag{5}$$

where $d = w_L w_M$ and $M = L \cup \{r\}$. This sets up an action of the free monoid S^* on the set of all paths. We denote by

$$[L; s, t, \dots] = (L; s, t, \dots).S^* \tag{6}$$

the orbit of the path $(L; s, t, \dots)$ under this action. Such an orbit is called a street. The subspace spanned by the streets is in fact [8, Theorem 6.6] a subalgebra Ξ of A.

For a path $(L; s, t, \dots)$ of positive length, we define

$$\delta(L; s, t, \dots) = (L_s; t, \dots) - (L_s; t, \dots).s \tag{7}$$

and

$$\Delta(L; s, t, \dots) = \delta^l(L; s, t, \dots), \tag{8}$$

if the path $(\delta(L; s, t, \dots)$ has length l. Then Δ maps A into A_0, the subspace of paths of length 0. If A_0 is identified with $\Sigma(W)$ via $L \mapsto e_L$ then the restriction of Δ to Ξ is a surjective anti-homomorphism from Ξ to $\Sigma(W)$ thanks to [8, Theorem 9.5]. Moreover, the vertex set Λ of the quiver of $\Sigma(W)$ is the set of S^*-orbits on $\mathcal{P}(S)$, the paths of length 0, and the edges of the quiver are images under Δ of streets $[L; s, t, \dots]$.

An algorithm which selects a suitable subset of streets as images of the edges of the quiver and expresses relations in terms of this selection has been formulated [8] and implemented in the ZigZag package [9]. In each case this selection provides one possible way to identify the generators of the path algebra with specific elements of the descent algebra $\Sigma(W)$. In the following sections, the results of running this algorithm on particular Coxeter groups are presented in the form of diagrams and tables. There we will use streets to label vertices and edges of quivers.

1.4 Notation.

In order to keep the descriptions of quivers short, we use various notational conventions for dealing with parabolic subgroups, subsets of S and the edges of the quiver.

The letter S is reserved for the set of Coxeter generators of W, which we identify with the integers $\{1, 2, \dots, n\}$ for $n = |S|$. Since $n < 9$ in the remaining sections, we can omit curly braces around and commas between elements of S in the tables below. E.g, the street $[\{1, 2, 3, 5, 6\}; 1, 6]$ will be written as $[12356; 16]$. Also, for $i \in S$, we use S_i as a shorthand for $S \setminus \{i\}$.

We denote the conjugacy class of the parabolic subgroup W_J by the isomorphism type of W_J. A name $X_{jkl\dots}$ denotes a class of parabolic subgroups of type $X_j \times A_k \times A_l \times \dots$. (This naming convention covers all subgroups of an irreducible finite Coxeter group W, since a parabolic subgroup of W is a direct product of irreducible finite Coxeter groups with at most one factor not of type A.) In case there are several classes of parabolic subgroups of the same type X_{jkl}, we use the primed names X'_{jkl}, X''_{jkl}, …, in order to distinguish them.

In a quiver, multiple edges between the same two vertices will be distinguished by using dotted arrows $\dot{\to}$, $\ddot{\to}$, …

2 Type $I_2(m)$.

The Coxeter group W of type $I_2(m)$, $m \geqslant 3$, has Coxeter diagram:

$$1 \overset{m}{-} 2$$

The structure of the descent algebra of W depends on whether m is even or odd. The longest element w_0 is central in W if and only if m is even. In any case, the structure of these 4-dimensional algebras is not particularly complicated. For completeness, we show the details here in two columns, on the left for m even and on the right for m odd.

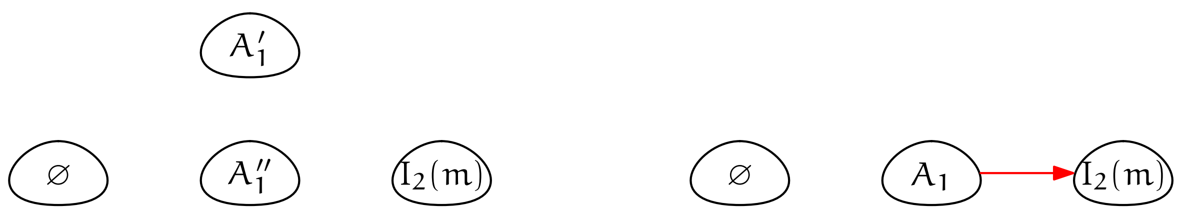

Figure 1: The quiver of type $I_2(m)$, m even (left) and m odd (right).

Quiver. The quiver of the descent algebra $\Sigma(W)$, as shown in Figure 1, has 4 vertices and no edges, if m is even.

v	type	λ
1.	$\varnothing$	$[\varnothing]$
2.	A_1'	$[1]$
3.	A_1''	$[2]$
4.	$I_2(m)$	$[12]$

Quiver. The quiver of the descent algebra $\Sigma(W)$, as shown in Figure 1, has 3 vertices and 1 edge, if m is odd.

v	type	λ
1.	$\varnothing$	$[\varnothing]$
2.	A_1	$[1]$
3.	$I_2(m)$	$[12]$

e	α
$2 \to 3.$	$[12; 1]$

Relations. There are no relations. These descent algebras are path algebras.

Projectives.

$\boxed{\varnothing}$, $\boxed{A_1'}$, $\boxed{A_1''}$, $\boxed{I_2(m)}$, if m is even.

Projectives.

$\boxed{\varnothing}$, $\boxed{A_1}$, $\begin{array}{|c|}\hline I_2(m) \\ \hline A_1 \\ \hline\end{array}$, if m is odd.

Cartan Matrix.

$\varnothing$	1	.
$I_2(m)$	.	1

A_1'	1	.
A_1''	.	1

Cartan Matrix.

$\varnothing$	1	.	.
A_1	.	1	.
$I_2(m)$	.	1	1

3 Type H_3.

The Coxeter group W of type H_3 has Coxeter diagram:

$$1 \overset{5}{-} 2 - 3$$

In this group, the longest element w_0 is central.

Quiver. The quiver has 6 vertices and 2 edges.

v	type	λ
1.	$\varnothing$	$[\varnothing]$
3.	A_{11}	[13]
4.	A_2	[23]
5.	H_2	[12]

v	type	λ
2.	A_1	[1]
6.	H_3	[S]

e	α
$2 \overset{\cdot}{\dashrightarrow} 6.$	[123; 12]
$2 \overset{\cdot\cdot}{\dashrightarrow} 6.$	[123; 31]

Figure 2 shows the quiver.

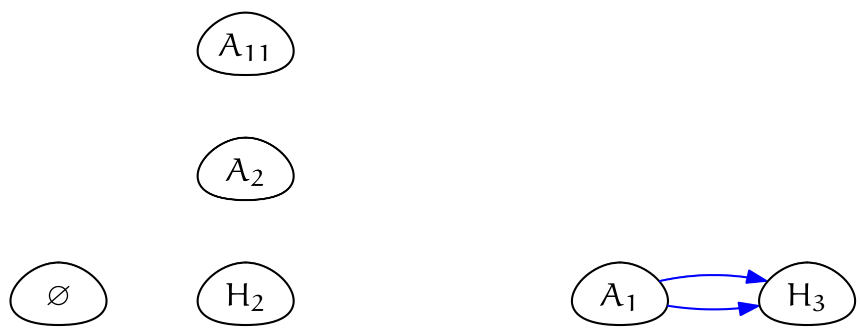

Figure 2: The quiver of type H_3.

Relations. There are no relations. This descent algebra is a path algebra.

Projectives.

$\boxed{\varnothing}$, $\boxed{\mathsf{A}_1}$, $\boxed{\mathsf{A}_{11}}$, $\boxed{\mathsf{A}_2}$, $\boxed{\mathsf{H}_2}$, $\begin{array}{|c|}\hline \mathsf{H}_3 \\ \hline (\mathsf{A}_1)^2 \\ \hline \end{array}$.

Cartan Matrix.

$\varnothing$	1	.	.	.
A_{11}	.	1	.	.
A_2	.	.	1	.
H_2	.	.	.	1

A_1	1	.
H_3	2	1

4 Type H_4.

The Coxeter group W of type H_4 has Coxeter diagram:

$$1 \overset{5}{-} 2 - 3 - 4$$

In this group, the longest element w_0 is central.

Quiver. The quiver, as shown in Figure 3, has 10 vertices and 6 edges. Note that the even part is dual to the odd part.

v	type	λ
1.	$\varnothing$	$[\varnothing]$
3.	A_{11}	[13]
4.	A_2	[23]
5.	H_2	[12]
10.	H_4	[S]

e	α
$3 \overset{.}{\to} 10.$	[S; 23]
$3 \overset{..}{\to} 10.$	[S; 31]
$4 \to 10.$	[S; 12]

v	type	λ
2.	A_1	[1]
6.	A_{21}	[134]
7.	H_{21}	[124]
8.	A_3	[234]
9.	H_3	[123]

e	α
$2 \to 8.$	[234; 23]
$2 \overset{.}{\to} 9.$	[123; 12]
$2 \overset{..}{\to} 9.$	[123; 31]

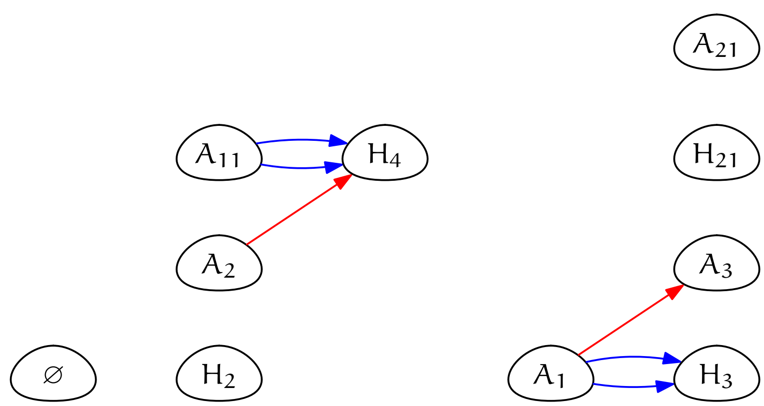

Figure 3: The quiver of type H_4.

Relations. There are no relations. This descent algebra is a path algebra.

Projectives.

$\boxed{\varnothing}$, $\boxed{A_1}$, $\boxed{A_{11}}$, $\boxed{A_2}$, $\boxed{H_2}$, $\boxed{A_{21}}$, $\boxed{H_{21}}$, $\begin{array}{|c|}\hline A_3 \\ \hline A_1 \\ \hline\end{array}$, $\begin{array}{|c|}\hline H_3 \\ \hline (A_1)^2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline H_4 \\ \hline (A_{11})^2\, A_2 \\ \hline\end{array}$.

Cartan Matrix.

$\varnothing$	1	.	.	.	.
A_{11}	.	1	.	.	.
A_2	.	.	1	.	.
H_2	.	.	.	1	.
H_4	.	2	1	.	1

A_1	1	.	.	.	.
A_{21}	.	1	.	.	.
H_{21}	.	.	1	.	.
A_3	1	.	.	1	.
H_3	2	.	.	.	1

5 Type F_4.

The Coxeter group W of type F_4 has Coxeter diagram:

$$1 - 2 = 3 - 4$$

In this group, the longest element w_0 is central.

Quiver. The quiver has 12 vertices and 4 edges.

v	type	λ
1.	$\varnothing$	$[\varnothing]$
4.	A_{11}	[13]
5.	A_2'	[12]
6.	A_2''	[34]
7.	B_2	[23]
12.	F_4	[S]

e	α
$4 \dashrightarrow 12.$	[S; 23]
$4 \overset{..}{\rightarrow} 12.$	[S; 31]

v	type	λ
2.	A_1'	[1]
3.	A_1''	[3]
8.	A_{21}'	[124]
9.	A_{21}''	[134]
10.	B_3'	[123]
11.	B_3''	[234]

e	α
$2 \rightarrow 10.$	[123; 31]
$3 \rightarrow 11.$	[234; 23]

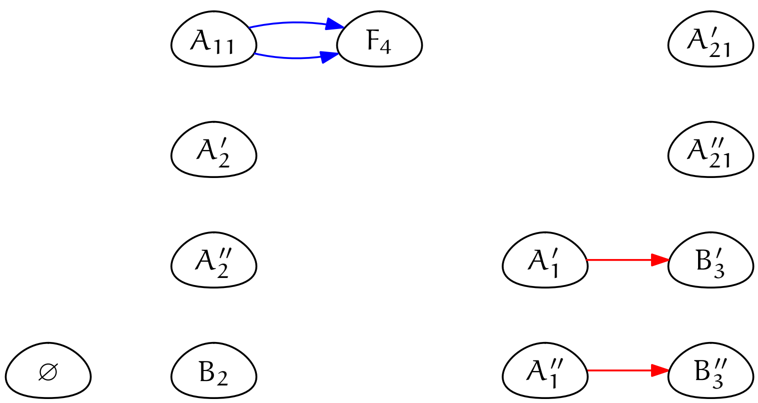

Figure 4: The quiver of type F_4.

Relations. There are no relations. This descent algebra is a path algebra.

Projectives.

$\boxed{\varnothing}$, $\boxed{A_1'}$, $\boxed{A_1''}$, $\boxed{A_{11}}$, $\boxed{A_2'}$, $\boxed{A_2''}$, $\boxed{B_2}$, $\boxed{A_{21}'}$, $\boxed{A_{21}''}$, $\begin{array}{|c|}\hline B_3' \\ \hline A_1' \\ \hline\end{array}$, $\begin{array}{|c|}\hline B_3'' \\ \hline A_1'' \\ \hline\end{array}$, $\begin{array}{|c|}\hline F_4 \\ \hline (A_{11})^2 \\ \hline\end{array}$.

Cartan Matrix.

$\varnothing$	1	.	.	.	.	.
A_{11}	.	1	.	.	.	.
A_2'	.	.	1	.	.	.
A_2''	.	.	.	1	.	.
B_2	.	.	.	.	1	.
F_4	.	2	.	.	.	1

A_1'	1	.	.	.	.	.
A_1''	.	1	.	.	.	.
A_{21}'	.	.	1	.	.	.
A_{21}''	.	.	.	1	.	.
B_3'	1	.	.	.	1	.
B_3''	.	1	.	.	.	1

6 Type E_6.

The Coxeter group W of type E_6 has Coxeter diagram:

$$\begin{array}{c} \quad\quad\; 2 \\ \quad\quad\; | \\ 1 - 3 - 4 - 5 - 6 \end{array}$$

Quiver. The quiver, as shown in Figure 5, has 17 vertices and 19 edges.

v	type	λ	v	type	λ
1.	∅	[∅]	10.	A_{31}	[1245]
2.	A_1	[1]	11.	A_4	[1345]
3.	A_{11}	[12]	12.	D_4	[2345]
4.	A_2	[13]	13.	A_{221}	$[S_4]$
5.	A_{111}	[146]	14.	A_{41}	$[S_5]$
6.	A_{21}	[124]	15.	A_5	$[S_2]$
7.	A_3	[134]	16.	D_5	$[S_6]$
8.	A_{211}	[1246]	17.	E_6	[S]
9.	A_{22}	[1356]			

e	α	e	α
$2 \to 7.$	[134; 13]	$8 \to 13.$	$[S_4; 1]$
$3 \to 6.$	[123; 1]	$8 \to 14.$	$[S_5; 3]$
$4 \to 11.$	[1234; 13]	$8 \to 17.$	[S; 41]
$5 \to 13.$	$[S_4; 16]$	$10 \to 14.$	$[S_5; 1]$
$5 \to 14.$	$[S_5; 34]$	$10 \to 15.$	$[S_2; 3]$
$5 \to 16.$	$[S_6; 41]$	$11 \to 15.$	$[S_2; 1]$
$6 \to 9.$	[1356; 1]	$11 \to 16.$	$[S_6; 2]$
$6 \to 10.$	[1245; 2]	$14 \to 17.$	[S; 3]
$6 \to 11.$	[1234; 3]	$16 \to 17.$	[S; 1]
$7 \to 11.$	[1234; 1]		

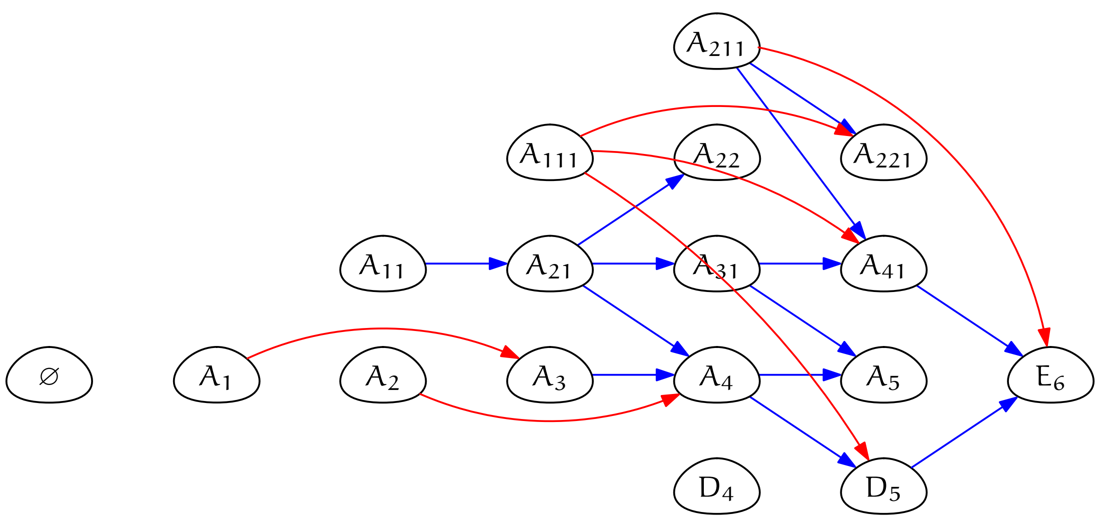

Figure 5: The quiver of type E_6.

Relations. There are two relations, one on paths of length 2 and one on paths of length 3:

$$(5 \to 16 \to 17) = -2(5 \to 14 \to 17),$$
$$(3 \to 6 \to 11 \to 15) = -(3 \to 6 \to 10 \to 15).$$

Projectives.

								A_{22}	A_{31}	A_4	
					A_{21}	A_3		A_{21}	A_{21}	$A_2\,A_{21}\,A_3$	
$\varnothing$,	A_1,	A_{11},	A_2,	A_{111},	A_{11},	A_1,	A_{211},	A_{11},	A_{11},	$A_1\,A_{11}$,	D_4,

				E_6
	A_{41}	A_5	D_5	$A_{211}\,A_{41}\,D_5$
	$A_{111}\,A_{211}\,A_{31}$	$A_{31}\,A_4$	$A_{111}\,A_4$	$A_{111}\,A_{211}\,A_{31}\,A_4$
A_{221}	A_{21}	$A_2\,(A_{21})^2\,A_3$	$A_2\,A_{21}\,A_3$	$A_2\,(A_{21})^2\,A_3$
$A_{111}\,A_{211}$,	A_{11},	$A_1\,A_{11}$,	$A_1\,A_{11}$,	$A_1\,(A_{11})^2$.

Note that the projective module E_6 contains copies of the simple module A_{211} in the second and the third layer of its Loewy series. These correspond to the images under Δ of the paths $[S;41]$ and $[S;3] \circ [S_5;3]$.

Cartan Matrix.

$\varnothing$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{11}	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_2	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{111}	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
A_{21}	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A_3	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
A_{211}	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
A_{22}	.	.	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.
A_{31}	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.
A_4	.	1	1	1	.	1	1	.	.	.	1	.	.	.	.	.	.
D_4	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
A_{221}	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.	.	.
A_{41}	.	.	1	.	1	1	.	1	.	1	.	.	.	1	.	.	.
A_5	.	1	1	1	.	2	1	.	.	1	1	.	.	.	1	.	.
D_5	.	1	1	1	1	1	1	.	.	.	1	.	.	.	.	1	.
E_6	.	1	2	1	1	2	1	2	.	1	1	.	.	1	.	1	1

7 Type E_7.

The Coxeter group W of type E_7 has Coxeter diagram:

$$
\begin{array}{ccccccccccccc}
 & & & & 2 & & & & & & & & \\
 & & & & | & & & & & & & & \\
1 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7
\end{array}
$$

In this group, the longest element w_0 is central.

Quiver. There are 32 vertices and 62 edges in total. The quiver of the even part, as shown in Figure 6, has 17 vertices and 33 edges.

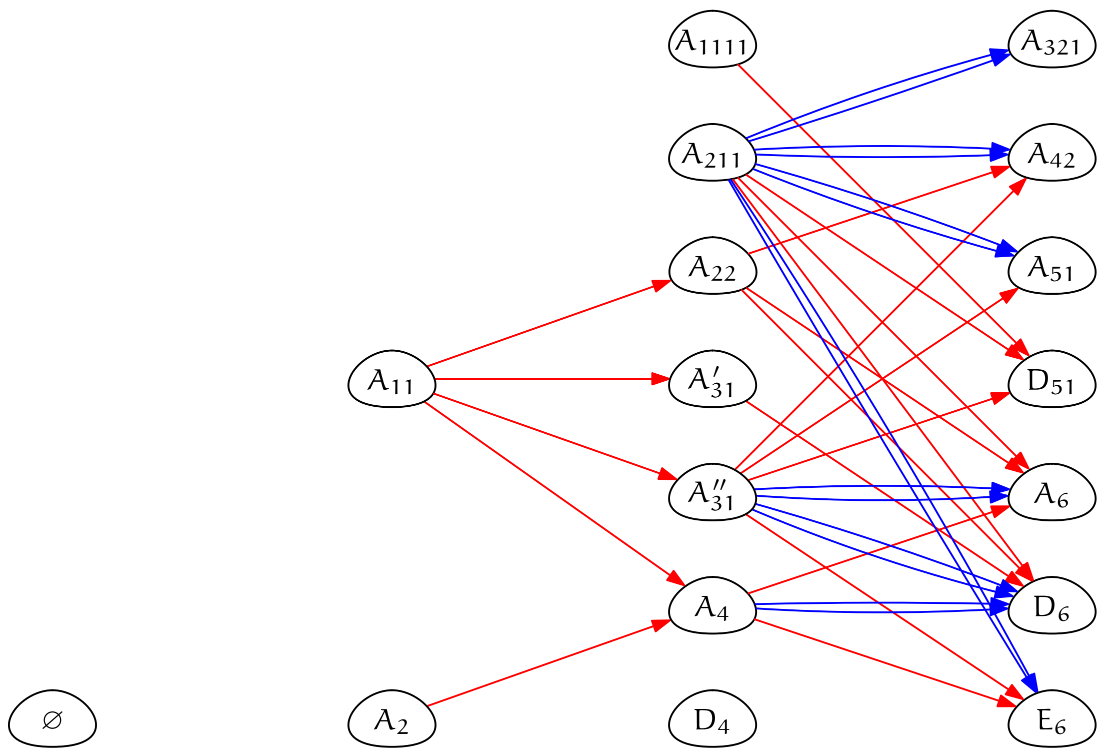

Figure 6: The even part of the quiver of type E_7.

v	type	λ	**v**	type	λ	**v**	type	λ	**v**	type	λ
1.	$\varnothing$	$[\varnothing]$	11.	A_{22}	[2467]	25.	A_{321}	$[S_4]$	30.	D_6	$[S_1]$
3.	A_{11}	[12]	12.	A'_{31}	[2457]	26.	A_{42}	$[S_5]$	31.	E_6	$[S_7]$
4.	A_2	[24]	13.	A''_{31}	[1245]	27.	A_{51}	$[S_3]$			
9.	A_{1111}	[1257]	14.	A_4	[1345]	28.	D_{51}	$[S_6]$			
10.	A_{211}	[1235]	15.	D_4	[2345]	29.	A_6	$[S_2]$			

e	α	e	α	e	α	e	α
$3 \to 11.$	$[1356; 15]$	$10 \ddot{\to} 26.$	$[S_5; 36]$	$11 \to 29.$	$[S_2; 45]$	$13 \ddot{\to} 30.$	$[S_1; 56]$
$3 \to 12.$	$[2457; 24]$	$10 \dot{\to} 27.$	$[S_3; 45]$	$11 \to 30.$	$[S_1; 52]$	$13 \to 31.$	$[S_7; 32]$
$3 \to 13.$	$[1245; 24]$	$10 \ddot{\to} 27.$	$[S_3; 52]$	$12 \to 30.$	$[S_1; 34]$	$14 \to 29.$	$[S_2; 13]$
$3 \to 14.$	$[1234; 32]$	$10 \to 28.$	$[S_6; 23]$	$13 \to 26.$	$[S_5; 16]$	$14 \dot{\to} 30.$	$[S_1; 23]$
$4 \to 14.$	$[1234; 12]$	$10 \to 29.$	$[S_2; 35]$	$13 \to 27.$	$[S_3; 24]$	$14 \ddot{\to} 30.$	$[S_1; 32]$
$9 \to 28.$	$[S_6; 41]$	$10 \to 30.$	$[S_1; 45]$	$13 \to 28.$	$[S_6; 21]$	$14 \to 31.$	$[S_7; 12]$
$10 \dot{\to} 25.$	$[S_4; 15]$	$10 \dot{\to} 31.$	$[S_7; 34]$	$13 \dot{\to} 29.$	$[S_2; 14]$		
$10 \ddot{\to} 25.$	$[S_4; 56]$	$10 \ddot{\to} 31.$	$[S_7; 41]$	$13 \ddot{\to} 29.$	$[S_2; 41]$		
$10 \dot{\to} 26.$	$[S_5; 32]$	$11 \to 26.$	$[S_5; 12]$	$13 \dot{\to} 30.$	$[S_1; 24]$		

The quiver of the odd part, as shown in Figure 7, has 15 vertices and 29 edges.

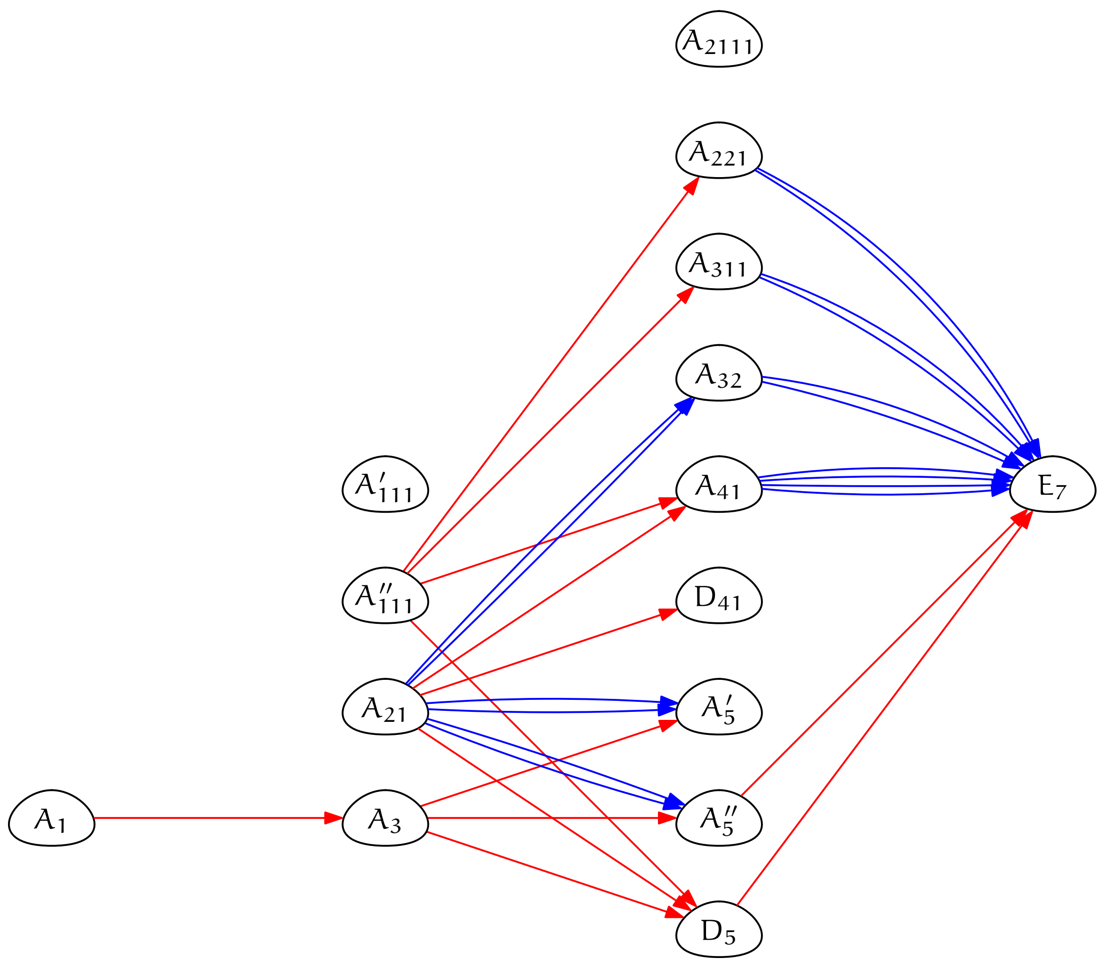

Figure 7: The odd part of the quiver of type E_7.

v	type	λ	v	type	λ	v	type	λ	v	type	λ
2.	A_1	[1]	8.	A_3	[234]	19.	A_{32}	[13467]	23.	A''_5	[34567]
5.	A'_{111}	[257]	16.	A_{2111}	[12357]	20.	A_{41}	[12347]	24.	D_5	[23456]
6.	A''_{111}	[127]	17.	A_{221}	[12367]	21.	D_{41}	[23457]	32.	E_7	[S]
7.	A_{21}	[123]	18.	A_{311}	[23567]	22.	A'_5	[24567]			

e	α	e	α	e	α	e	α
$2 \to 8.$	[134; 13]	$7 \to 21.$	[23457; 23]	$8 \to 24.$	[12345; 21]	$20 \overset{\cdot\cdot}{\to} 32.$	[S; 32]
$6 \to 17.$	[12356; 15]	$7 \overset{\cdot}{\to} 22.$	[24567; 45]	$17 \overset{\cdot}{\to} 32.$	[S; 45]	$20 \overset{\cdot\cdot\cdot}{\to} 32.$	[S; 56]
$6 \to 18.$	[12457; 24]	$7 \overset{\cdot\cdot}{\to} 22.$	[24567; 52]	$17 \overset{\cdot\cdot}{\to} 32.$	[S; 53]	$20 \overset{\cdot\cdot\cdot\cdot}{\to} 32.$	[S; 62]
$6 \to 20.$	[12346; 32]	$7 \overset{\cdot}{\to} 23.$	[13456; 34]	$18 \overset{\cdot}{\to} 32.$	[S; 34]	$23 \to 32.$	[S; 21]
$6 \to 24.$	[12345; 41]	$7 \overset{\cdot\cdot}{\to} 23.$	[13456; 41]	$18 \overset{\cdot\cdot}{\to} 32.$	[S; 41]	$24 \to 32.$	[S; 71]
$7 \overset{\cdot}{\to} 19.$	[13467; 13]	$7 \to 24.$	[12345; 23]	$19 \overset{\cdot}{\to} 32.$	[S; 24]		
$7 \overset{\cdot\cdot}{\to} 19.$	[13467; 16]	$8 \to 22.$	[24567; 24]	$19 \overset{\cdot\cdot}{\to} 32.$	[S; 51]		
$7 \to 20.$	[12346; 12]	$8 \to 23.$	[13456; 13]	$20 \overset{\cdot}{\to} 32.$	[S; 23]		

Relations. The presentation needs 13 relations. There are 6 relations on the even part:

$$(3 \to 13 \to 26) = -\tfrac{1}{2}(3 \to 11 \to 26), \qquad (3 \to 13 \overset{\cdot\cdot}{\to} 29) = \tfrac{1}{2}(3 \to 11 \to 29),$$
$$(3 \to 14 \to 29) = -(3 \to 13 \overset{\cdot}{\to} 29), \qquad (3 \to 13 \overset{\cdot\cdot}{\to} 30) = -\tfrac{1}{2}(3 \to 11 \to 30),$$
$$(3 \to 14 \overset{\cdot}{\to} 30) = -(3 \to 13 \overset{\cdot}{\to} 30), \qquad (3 \to 14 \overset{\cdot\cdot}{\to} 30) = -(3 \to 12 \to 30).$$

And there are 7 relations on the odd part:

$$(6 \to 18 \overset{\cdot\cdot}{\to} 32) = \tfrac{1}{2}(6 \to 17 \overset{\cdot}{\to} 32),$$
$$(6 \to 20 \overset{\cdot\cdot}{\to} 32) = -(6 \to 18 \overset{\cdot}{\to} 32),$$
$$(6 \to 20 \overset{\cdot\cdot\cdot}{\to} 32) = -\tfrac{1}{2}(6 \to 17 \overset{\cdot\cdot}{\to} 32),$$
$$(6 \to 24 \to 32) = (6 \to 20 \overset{\cdot\cdot\cdot\cdot}{\to} 32) + (6 \to 20 \overset{\cdot\cdot\cdot}{\to} 32) - (6 \to 20 \overset{\cdot\cdot}{\to} 32) + (6 \to 20 \overset{\cdot}{\to} 32),$$
$$(7 \to 20 \overset{\cdot\cdot\cdot}{\to} 32) = -(7 \overset{\cdot\cdot}{\to} 19 \overset{\cdot\cdot}{\to} 32),$$
$$(7 \overset{\cdot}{\to} 23 \to 32) = (7 \overset{\cdot\cdot}{\to} 19 \overset{\cdot}{\to} 32) - (7 \to 20 \overset{\cdot}{\to} 32),$$
$$(7 \overset{\cdot\cdot}{\to} 23 \to 32) = (7 \overset{\cdot\cdot}{\to} 19 \overset{\cdot}{\to} 32) - (7 \overset{\cdot}{\to} 19 \overset{\cdot}{\to} 32).$$

Projectives.

$$\boxed{\varnothing},\ \boxed{A_1},\ \boxed{A_{11}},\ \boxed{A_2},\ \boxed{A'_{111}},\ \boxed{A''_{111}},\ \boxed{A_{21}},\ \begin{array}{|c|}\hline A_3\\\hline A_1\\\hline\end{array},\ \boxed{A_{1111}},\ \boxed{A_{211}},\ \begin{array}{|c|}\hline A_{22}\\\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline A'_{31}\\\hline A_{11}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline A''_{31}\\\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline A_4\\\hline A_{11}\,A_2\\\hline\end{array},\ \boxed{D_4},\ \boxed{A_{2111}},\ \begin{array}{|c|}\hline A_{221}\\\hline A''_{111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{311}\\\hline A''_{111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{32}\\\hline (A_{21})^2\\\hline\end{array},\ \begin{array}{|c|}\hline A_{41}\\\hline A''_{111}\,A_{21}\\\hline\end{array},\ \begin{array}{|c|}\hline D_{41}\\\hline A_{21}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline A'_5\\\hline (A_{21})^2\,A_3\\\hline A_1\\\hline\end{array},\ \begin{array}{|c|}\hline A''_5\\\hline (A_{21})^2\,A_3\\\hline A_1\\\hline\end{array},\ \begin{array}{|c|}\hline D_5\\\hline A''_{111}\,A_{21}\,A_3\\\hline A_1\\\hline\end{array},\ \begin{array}{|c|}\hline A_{321}\\\hline (A_{211})^2\\\hline\end{array},\ \begin{array}{|c|}\hline A_{42}\\\hline (A_{211})^2\,A_{22}\,A''_{31}\\\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{51}\\\hline (A_{211})^2\,A''_{31}\\\hline A_{11}\\\hline\end{array},$$

D_{51}
$A_{1111}\ A_{211}\ A''_{31}$
A_{11}

,

A_6
$A_{211}\ A_{22}\ (A''_{31})^2\ A_4$
$(A_{11})^2\ A_2$

,

D_6
$A_{211}\ A_{22}\ A'_{31}\ (A''_{31})^2\ (A_4)^2$
$(A_{11})^3\ (A_2)^2$

,

E_6
$(A_{211})^2\ A''_{31}\ A_4$
$(A_{11})^2\ A_2$

,

E_7
$(A_{221})^2\ (A_{311})^2\ (A_{32})^2\ (A_{41})^4\ A''_5\ D_5$
$(A''_{111})^5\ (A_{21})^8\ (A_3)^2$
$(A_1)^2$

.

Cartan Matrix.

$\varnothing$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{11}	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_2	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{1111}	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{211}	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
A_{22}	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A'_{31}	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
A''_{31}	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
A_4	.	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.
D_4	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
A_{321}	.	.	.	.	2	.	.	.	.	.	1	.	.	.	.	.	.
A_{42}	.	1	.	.	2	1	.	1	.	.	.	1	.	.	.	.	.
A_{51}	.	1	.	.	2	.	.	1	.	.	.	.	1	.	.	.	.
D_{51}	.	1	.	1	1	.	.	1	.	.	.	.	.	1	.	.	.
A_6	.	2	1	.	1	1	.	2	1	.	.	.	.	.	1	.	.
D_6	.	3	2	.	1	1	1	2	2	.	.	.	.	.	.	1	.
E_6	.	2	1	.	2	.	.	1	1	.	.	.	.	.	.	.	1

A_1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A'_{111}	.	1	.	.	.	.	.	.	.	.	.	.	.	.
A''_{111}	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A_{21}	.	.	.	1	.	.	.	.	.	.	.	.	.	.
A_3	1	.	.	.	1	.	.	.	.	.	.	.	.	.
A_{2111}	.	.	.	.	.	1	.	.	.	.	.	.	.	.
A_{221}	.	.	1	.	.	.	1	.	.	.	.	.	.	.
A_{311}	.	.	1	.	.	.	.	1	.	.	.	.	.	.
A_{32}	.	.	.	2	.	.	.	.	1	.	.	.	.	.
A_{41}	.	.	1	1	.	.	.	.	.	1	.	.	.	.
D_{41}	.	.	.	1	.	.	.	.	.	.	1	.	.	.
A'_5	1	.	.	2	1	.	.	.	.	.	.	1	.	.
A''_5	1	.	.	2	1	.	.	.	.	.	.	.	1	.
D_5	1	.	1	1	1	.	.	.	.	.	.	.	.	1
E_7	2	.	5	8	2	.	2	2	2	4	.	.	1	1	1

8 Type E_8.

The Coxeter group W of type E_8 has Coxeter diagram:

$$\begin{array}{ccccccccccccc} & & & & 2 & & & & & & & & \\ & & & & | & & & & & & & & \\ 1 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 & - & 8 \end{array}$$

In this group, the longest element w_0 is central.

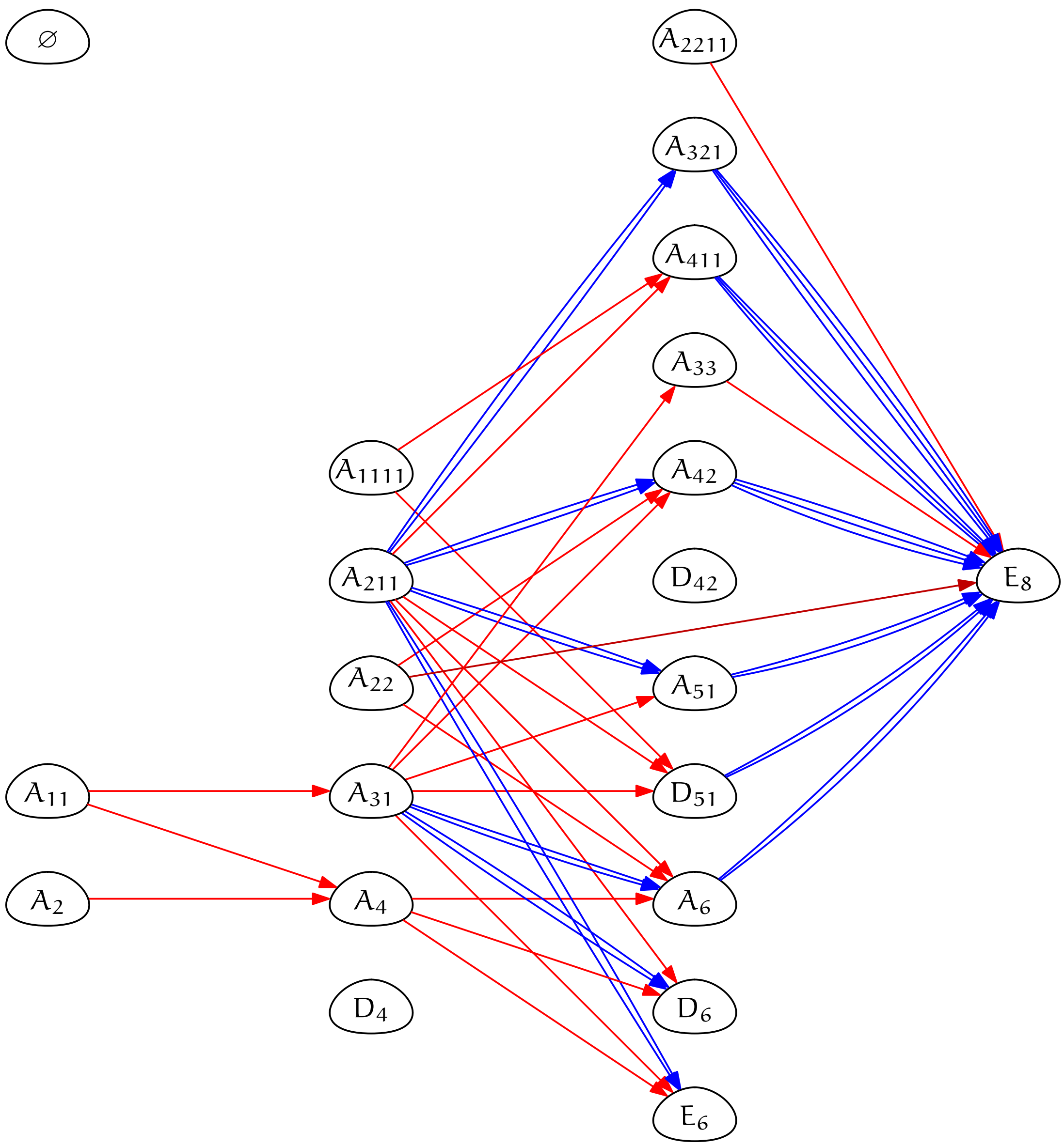

Figure 8: The even part of the quiver of type E_8.

Quiver and Relations. There are 41 vertices and 109 edges in total. The presentation needs 33 relations.

The quiver of the even part, as shown in Figure 8, has 21 vertices and 49 edges.

v	type	λ	**v**	type	λ	**v**	type	λ	**v**	type	λ
1.	$\varnothing$	$[\varnothing]$	11.	A_{31}	[1348]	25.	A_{33}	[134678]	31.	D_6	[234567]
3.	A_{11}	[12]	12.	A_4	[4567]	26.	A_{42}	[123467]	32.	E_6	[123456]
4.	A_2	[13]	13.	D_4	[2345]	27.	D_{42}	[234578]	41.	E_8	[S]
8.	A_{1111}	[1268]	22.	A_{2211}	[123578]	28.	A_{51}	[124567]			
9.	A_{211}	[2378]	23.	A_{321}	[123678]	29.	D_{51}	[123458]			
10.	A_{22}	[1367]	24.	A_{411}	[125678]	30.	A_6	[134567]			

e	α	**e**	α	**e**	α	**e**	α
$3 \to 11.$	[1245; 24]	$9 \to 30.$	[134567; 35]	$11 \dot{\to} 31.$	[234567; 24]	$25 \to 41.$	[S; 51]
$3 \to 12.$	[1234; 32]	$9 \to 31.$	[234567; 45]	$11 \ddot{\to} 31.$	[234567; 56]	$26 \dot{\to} 41.$	[S; 24]
$4 \to 12.$	[1234; 12]	$9 \dot{\to} 32.$	[123456; 34]	$11 \to 32.$	[123456; 32]	$26 \ddot{\to} 41.$	[S; 56]
$8 \to 24.$	[123468; 32]	$9 \ddot{\to} 32.$	[123456; 41]	$12 \to 30.$	[134567; 13]	$26 \dddot{\to} 41.$	[S; 62]
$8 \to 29.$	[123457; 41]	$10 \to 26.$	[123467; 12]	$12 \to 31.$	[234567; 23]	$28 \dot{\to} 41.$	[S; 23]
$9 \dot{\to} 23.$	[123567; 15]	$10 \to 30.$	[134567; 45]	$12 \to 32.$	[123456; 12]	$28 \ddot{\to} 41.$	[S; 32]
$9 \ddot{\to} 23.$	[123567; 56]	$10 \to 41.$	[S; 1652]	$22 \to 41.$	[S; 46]	$29 \dot{\to} 41.$	[S; 67]
$9 \to 24.$	[123468; 12]	$11 \to 25.$	[134678; 13]	$23 \dot{\to} 41.$	[S; 35]	$29 \ddot{\to} 41.$	[S; 71]
$9 \dot{\to} 26.$	[123467; 32]	$11 \to 26.$	[123467; 16]	$23 \ddot{\to} 41.$	[S; 45]	$30 \dot{\to} 41.$	[S; 12]
$9 \ddot{\to} 26.$	[123467; 36]	$11 \to 28.$	[124567; 24]	$23 \dddot{\to} 41.$	[S; 53]	$30 \ddot{\to} 41.$	[S; 21]
$9 \dot{\to} 28.$	[124567; 45]	$11 \to 29.$	[123457; 21]	$24 \dot{\to} 41.$	[S; 34]		
$9 \ddot{\to} 28.$	[124567; 52]	$11 \dot{\to} 30.$	[134567; 14]	$24 \ddot{\to} 41.$	[S; 41]		
$9 \to 29.$	[123457; 23]	$11 \ddot{\to} 30.$	[134567; 41]	$24 \dddot{\to} 41.$	[S; 73]		

There are 16 relations on the even part, 14 between paths of length 2:

$$(3 \to 12 \to 30) = -(3 \to 11 \dot{\to} 30),$$
$$(3 \to 12 \to 31) = -(3 \to 11 \dot{\to} 31),$$
$$(8 \to 29 \ddot{\to} 41) = (8 \to 24 \dddot{\to} 41),$$
$$(9 \to 24 \ddot{\to} 41) = -(9 \dot{\to} 23 \ddot{\to} 41),$$
$$(9 \dot{\to} 26 \ddot{\to} 41) = -(9 \dot{\to} 23 \dddot{\to} 41),$$
$$(9 \ddot{\to} 26 \ddot{\to} 41) = (9 \ddot{\to} 23 \dddot{\to} 41),$$
$$(9 \ddot{\to} 28 \dot{\to} 41) = (9 \dot{\to} 26 \dot{\to} 41) - (9 \ddot{\to} 26 \dot{\to} 41),$$
$$(9 \dot{\to} 28 \ddot{\to} 41) = -(9 \dot{\to} 23 \dot{\to} 41) - (9 \to 24 \dot{\to} 41),$$
$$(9 \ddot{\to} 28 \ddot{\to} 41) = -(9 \dot{\to} 23 \dot{\to} 41) - (9 \ddot{\to} 23 \dot{\to} 41),$$
$$(9 \to 29 \dot{\to} 41) = -(9 \ddot{\to} 26 \dddot{\to} 41),$$
$$(9 \to 30 \ddot{\to} 41) = -(9 \ddot{\to} 26 \dot{\to} 41) + (9 \dot{\to} 28 \dot{\to} 41) + (9 \ddot{\to} 28 \dot{\to} 41),$$
$$(10 \to 30 \ddot{\to} 41) = -(10 \to 26 \dot{\to} 41),$$
$$(11 \to 29 \dot{\to} 41) = -(11 \to 26 \dddot{\to} 41),$$
$$(11 \ddot{\to} 30 \ddot{\to} 41) = (11 \to 26 \dot{\to} 41) - (11 \to 28 \dot{\to} 41) + (11 \dot{\to} 30 \ddot{\to} 41),$$

and two between paths of length 3 from vertex A_{11} to vertex E_8:

$$(3 \to 11 \to 26 \overset{\cdot\cdot}{\to} 41) = \tfrac{1}{2}(3 \to 11 \to 25 \to 41),$$
$$(3 \to 11 \overset{\cdot\cdot}{\to} 30 \overset{\cdot\cdot}{\to} 41) = -(3 \to 11 \to 26 \overset{\cdot}{\to} 41).$$

The quiver of the odd part, as shown in Figure 9, has 20 vertices and 60 edges.

v	type	λ	**v**	type	λ	**v**	type	λ	**v**	type	λ
2.	A_1	[1]	15.	A_{221}	[12367]	20.	A_5	[13456]	36.	D_{52}	$[S_6]$
5.	A_{111}	[147]	16.	A_{311}	[12458]	21.	D_5	[23456]	37.	A_7	$[S_2]$
6.	A_{21}	[124]	17.	A_{32}	[24578]	33.	A_{421}	$[S_4]$	38.	E_{61}	$[S_7]$
7.	A_3	[245]	18.	A_{41}	[24568]	34.	A_{43}	$[S_5]$	39.	D_7	$[S_1]$
14.	A_{2111}	[12568]	19.	D_{41}	[23458]	35.	A_{61}	$[S_3]$	40.	E_7	$[S_8]$

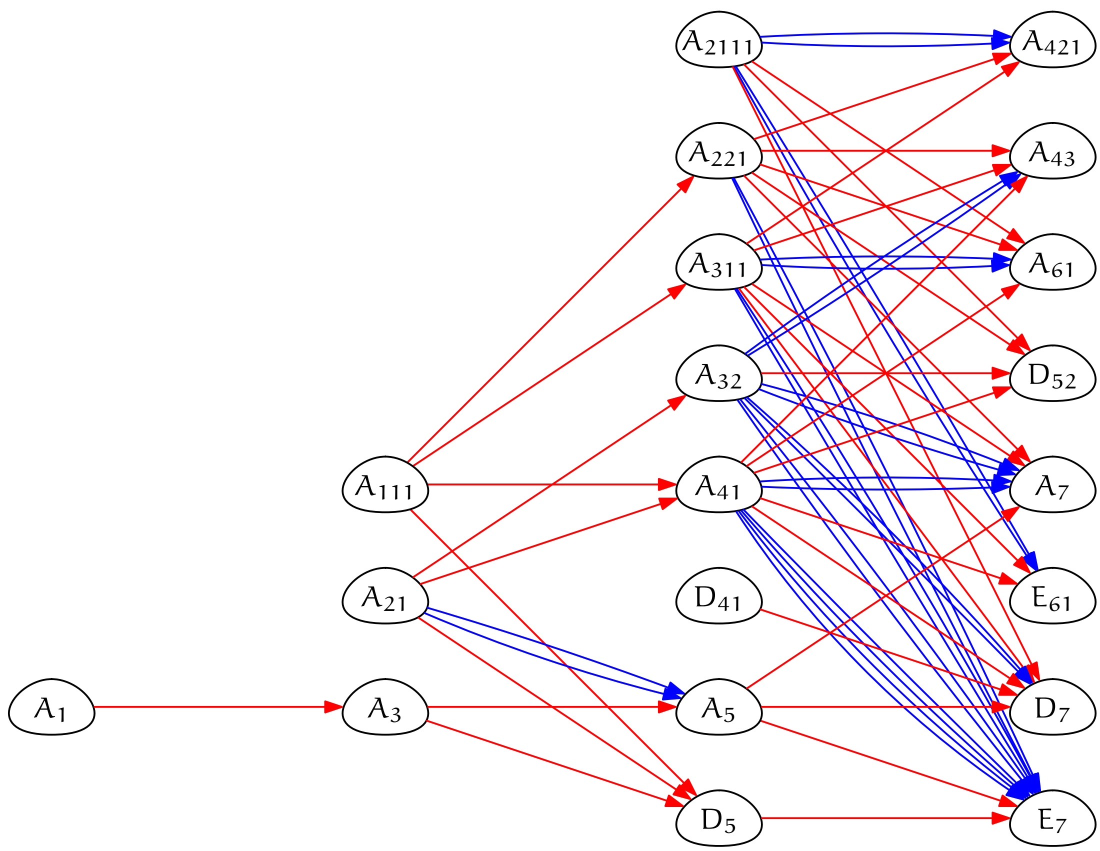

Figure 9: The odd part of the quiver of type E_8.

$\mathbf{e}$	α	$\mathbf{e}$	α	$\mathbf{e}$	α	$\mathbf{e}$	α
$2 \to 7.$	$[134; 13]$	$14 \to 36.$	$[S_6; 41]$	$16 \to 37.$	$[S_2; 35]$	$18 \to 35.$	$[S_3; 24]$
$5 \to 15.$	$[12356; 15]$	$14 \dot{\to} 38.$	$[S_7; 34]$	$16 \to 38.$	$[S_7; 32]$	$18 \to 36.$	$[S_6; 27]$
$5 \to 16.$	$[12457; 24]$	$14 \ddot{\to} 38.$	$[S_7; 41]$	$16 \to 39.$	$[S_1; 45]$	$18 \dot{\to} 37.$	$[S_2; 14]$
$5 \to 18.$	$[12346; 32]$	$14 \to 39.$	$[S_1; 46]$	$16 \dot{\to} 40.$	$[S_8; 34]$	$18 \ddot{\to} 37.$	$[S_2; 34]$
$5 \to 21.$	$[12345; 41]$	$15 \to 33.$	$[S_4; 56]$	$16 \ddot{\to} 40.$	$[S_8; 41]$	$18 \to 38.$	$[S_7; 12]$
$6 \to 17.$	$[13467; 13]$	$15 \to 34.$	$[S_5; 36]$	$17 \dot{\to} 34.$	$[S_5; 12]$	$18 \to 39.$	$[S_1; 24]$
$6 \to 18.$	$[12346; 12]$	$15 \to 35.$	$[S_3; 56]$	$17 \ddot{\to} 34.$	$[S_5; 16]$	$18 \dot{\to} 40.$	$[S_8; 23]$
$6 \dot{\to} 20.$	$[13456; 34]$	$15 \to 36.$	$[S_6; 23]$	$17 \to 36.$	$[S_6; 21]$	$18 \ddot{\to} 40.$	$[S_8; 32]$
$6 \ddot{\to} 20.$	$[13456; 41]$	$15 \to 37.$	$[S_2; 46]$	$17 \dot{\to} 37.$	$[S_2; 15]$	$18 \dddot{\to} 40.$	$[S_8; 56]$
$6 \to 21.$	$[12345; 23]$	$15 \dot{\to} 40.$	$[S_8; 45]$	$17 \ddot{\to} 37.$	$[S_2; 51]$	$18 \ddddot{\to} 40.$	$[S_8; 62]$
$7 \to 20.$	$[13456; 13]$	$15 \ddot{\to} 40.$	$[S_8; 53]$	$17 \dot{\to} 39.$	$[S_1; 25]$	$19 \to 39.$	$[S_1; 67]$
$7 \to 21.$	$[12345; 21]$	$16 \to 33.$	$[S_4; 51]$	$17 \ddot{\to} 39.$	$[S_1; 56]$	$20 \to 37.$	$[S_2; 13]$
$14 \dot{\to} 33.$	$[S_4; 61]$	$16 \to 34.$	$[S_5; 32]$	$17 \dot{\to} 40.$	$[S_8; 24]$	$20 \to 39.$	$[S_1; 23]$
$14 \ddot{\to} 33.$	$[S_4; 67]$	$16 \dot{\to} 35.$	$[S_3; 25]$	$17 \ddot{\to} 40.$	$[S_8; 51]$	$20 \to 40.$	$[S_8; 21]$
$14 \to 35.$	$[S_3; 46]$	$16 \ddot{\to} 35.$	$[S_3; 52]$	$18 \to 34.$	$[S_5; 67]$	$21 \to 40.$	$[S_8; 71]$

There are 17 relations on the odd part, all between paths of length 2:

$$(5 \to 16 \to 33) = \tfrac{1}{2}(5 \to 15 \to 33),$$
$$(5 \to 16 \to 34) = -\tfrac{1}{2}(5 \to 15 \to 34),$$
$$(5 \to 18 \to 34) = -\tfrac{1}{2}(5 \to 15 \to 34),$$
$$(5 \to 16 \ddot{\to} 35) = -\tfrac{1}{2}(5 \to 15 \to 35),$$
$$(5 \to 18 \to 35) = -(5 \to 16 \dot{\to} 35),$$
$$(5 \to 18 \to 36) = \tfrac{1}{2}(5 \to 15 \to 36),$$
$$(5 \to 18 \dot{\to} 37) = \tfrac{1}{2}(5 \to 15 \to 37) - (5 \to 16 \to 37),$$
$$(5 \to 18 \ddot{\to} 37) = -(5 \to 16 \to 37),$$
$$(5 \to 16 \ddot{\to} 40) = -\tfrac{1}{2}(5 \to 15 \dot{\to} 40),$$
$$(5 \to 18 \dddot{\to} 40) = -\tfrac{1}{2}(5 \to 15 \ddot{\to} 40),$$
$$(5 \to 18 \ddot{\to} 40) = -(5 \to 16 \to 40),$$
$$(5 \to 21 \to 40) = (5 \to 18 \ddddot{\to} 40) + (5 \to 18 \dddot{\to} 40) - (5 \to 18 \ddot{\to} 40) + (5 \to 18 \dot{\to} 40),$$
$$(6 \to 18 \to 34) = (6 \to 17 \dot{\to} 34),$$
$$(6 \ddot{\to} 20 \to 37) = -(6 \to 17 \dot{\to} 37) + (6 \to 18 \dot{\to} 37) + (6 \dot{\to} 20 \to 37),$$
$$(6 \dot{\to} 20 \to 37) = -(6 \to 17 \ddot{\to} 37) + (6 \to 18 \ddot{\to} 37) - 2(6 \to 18 \dot{\to} 37),$$
$$(6 \ddot{\to} 20 \to 39) = -(6 \to 17 \dot{\to} 39) + (6 \to 18 \to 39) + (6 \dot{\to} 20 \to 39),$$
$$(6 \ddot{\to} 20 \to 40) = -(6 \to 17 \dot{\to} 40) + (6 \to 18 \dot{\to} 40) + (6 \dot{\to} 20 \to 40).$$

Projectives.

$$\begin{array}{|c|}\hline \varnothing\\\hline\end{array},\ \begin{array}{|c|}\hline A_{1}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{2}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{21}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{3}\\\hline A_{1}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{1111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{211}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{22}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{31}\\\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{4}\\\hline A_{11}\,A_{2}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline D_{4}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{2111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{221}\\\hline A_{111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{311}\\\hline A_{111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{32}\\\hline A_{21}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{41}\\\hline A_{111}\,A_{21}\\\hline\end{array},\ \begin{array}{|c|}\hline D_{41}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{5}\\\hline (A_{21})^2\,A_{3}\\\hline A_{1}\\\hline\end{array},\ \begin{array}{|c|}\hline D_{5}\\\hline A_{111}\,A_{21}\,A_{3}\\\hline A_{1}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline A_{2211}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{321}\\\hline (A_{211})^2\\\hline\end{array},\ \begin{array}{|c|}\hline A_{411}\\\hline A_{1111}\,A_{211}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{33}\\\hline A_{31}\\\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{42}\\\hline (A_{211})^2\,A_{22}\,A_{31}\\\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline D_{42}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{51}\\\hline (A_{211})^2\,A_{31}\\\hline A_{11}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline D_{51}\\\hline A_{1111}\,A_{211}\,A_{31}\\\hline A_{11}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{6}\\\hline A_{211}\,A_{22}\,(A_{31})^2\,A_{4}\\\hline (A_{11})^2\,A_{2}\\\hline\end{array},\ \begin{array}{|c|}\hline D_{6}\\\hline A_{211}\,(A_{31})^2\,A_{4}\\\hline (A_{11})^2\,A_{2}\\\hline\end{array},\ \begin{array}{|c|}\hline E_{6}\\\hline (A_{211})^2\,A_{31}\,A_{4}\\\hline (A_{11})^2\,A_{2}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline A_{421}\\\hline (A_{2111})^2\,A_{221}\,A_{311}\\\hline A_{111}\\\hline\end{array},\ \begin{array}{|c|}\hline A_{43}\\\hline A_{221}\,A_{311}\,(A_{32})^2\,A_{41}\\\hline A_{111}\,(A_{21})^2\\\hline\end{array},\ \begin{array}{|c|}\hline A_{61}\\\hline A_{2111}\,A_{221}\,(A_{311})^2\,A_{41}\\\hline (A_{111})^2\,A_{21}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline D_{52}\\\hline A_{2111}\,A_{221}\,A_{32}\,A_{41}\\\hline A_{111}\,(A_{21})^2\\\hline\end{array},\ \begin{array}{|c|}\hline A_{7}\\\hline A_{221}\,A_{311}\,(A_{32})^2\,(A_{41})^2\,A_{5}\\\hline (A_{111})^2\,(A_{21})^4\,A_{3}\\\hline A_{1}\\\hline\end{array},\ \begin{array}{|c|}\hline E_{61}\\\hline (A_{2111})^2\,A_{311}\,A_{41}\\\hline (A_{111})^2\,A_{21}\\\hline\end{array},$$

$$\begin{array}{|c|}\hline D_{7}\\\hline A_{2111}\,A_{311}\,(A_{32})^2\,A_{41}\,D_{41}\,A_{5}\\\hline (A_{111})^2\,(A_{21})^4\,A_{3}\\\hline A_{1}\\\hline\end{array},\ \begin{array}{|c|}\hline E_{7}\\\hline (A_{221})^2\,(A_{311})^2\,(A_{32})^2\,(A_{41})^4\,A_{5}\,D_{5}\\\hline (A_{111})^5\,(A_{21})^8\,(A_{3})^2\\\hline (A_{1})^2\\\hline\end{array},$$

$$\begin{array}{|c|}\hline E_{8}\\\hline A_{22}\,A_{2211}\,(A_{321})^3\,(A_{411})^3\,A_{33}\,(A_{42})^3\,(A_{51})^2\,(D_{51})^2\,(A_{6})^2\\\hline (A_{1111})^4\,(A_{211})^{15}\,(A_{22})^4\,(A_{31})^{10}\,(A_{4})^2\\\hline (A_{11})^8\,(A_{2})^2\\\hline\end{array}.$$

Cartan Matrix.

∅	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{11}	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{2}	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{1111}	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{211}	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{22}	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{31}	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{4}	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
D_{4}	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
A_{2211}	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
A_{321}	.	.	.	.	2	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
A_{411}	.	.	.	1	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
A_{33}	.	1	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.
A_{42}	.	1	.	.	2	1	1	.	.	.	.	.	.	1	.	.	.	.	.	.
D_{42}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
A_{51}	.	1	.	.	2	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.
D_{51}	.	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.
A_{6}	.	2	1	.	1	1	2	1	.	.	.	.	.	.	.	.	.	1	.	.	.
D_{6}	.	2	1	.	1	.	2	1	.	.	.	.	.	.	.	.	.	.	1	.	.
E_{6}	.	2	1	.	2	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.
E_{8}	.	8	2	4	15	5	10	2	.	1	3	3	1	3	.	2	2	2	.	.	1

A_{1}	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{111}	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{21}	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{3}	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{2111}	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{221}	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{311}	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
A_{32}	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
A_{41}	.	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
D_{41}	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
A_{5}	1	.	2	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
D_{5}	1	1	1	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
A_{421}	.	1	.	.	2	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.
A_{43}	.	1	2	.	.	1	1	2	1	.	.	.	.	1	.	.	.	.	.	.
A_{61}	.	2	1	.	1	1	2	.	1	.	.	.	.	.	1	.	.	.	.	.
D_{52}	.	1	2	.	1	1	.	1	1	.	.	.	.	.	.	1	.	.	.	.
A_{7}	1	2	4	1	.	1	1	2	2	.	1	.	.	.	.	.	1	.	.	.
E_{61}	.	2	1	.	2	.	1	.	1	.	.	.	.	.	.	.	.	1	.	.
D_{7}	1	2	4	1	1	.	1	2	1	1	1	.	.	.	.	.	.	.	1	.
E_{7}	2	5	8	2	.	2	2	2	4	.	1	1	.	.	.	.	.	.	.	1

9 Concluding Remarks.

The quiver of the descent algebra of a Coxeter group of type A has been described by Schocker [11]. The quiver of the descent algebra of a Coxeter group of type B has recently been constructed by Saliola [10]. No attempts have been made to describe the relations in these cases. The quiver of the descent algebra of a Coxeter group of type D is not known in general. On the basis of the known results, and some experiments with descent algebras of type D, we can classify the cases where no relations are needed.

Suppose that W is an irreducible finite Coxeter group. Then:

- the descent algebra $\Sigma(W)$ is a path algebra only if W is of type A_n with $n \leqslant 4$, B_n with $n \leqslant 5$, D_n with $n \leqslant 5$, F_4, H_3, H_4, or $I_2(m)$;
- the descent algebra $\Sigma(W)$ is commutative only if W is of type A_1, B_2, or $I_2(m)$ with $m \geqslant 6$ even.

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