Week 12: Optimisation and Course Review.

MA161/MA1161: Semester 1 Calculus.

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Optimization.

We have seen:

- A function $f(x)$ has a **critical point** at $x = p$ if either $f'(p)$ does not exist, or $f'(p) = 0$.
- By Fermat's Theorem, a point p where $f'(p) = 0$ may correspond to a local minimum, or a local maximum (or neither).
- We would like to use information about the derivative $f'(x)$ (and $f''(x)$, the **second derivative**) to decide which is the case.
- If $f'(x) > 0$ on an interval, then $f(x)$ is **increasing** on that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is **decreasing** on that interval.

Now:

- We will formulate two tests, the **First Derivative Test** and the **Second Derivative Test**, which can help to decide whether a critical point corresponds to a local minimum or maximum.

Recall: The First Derivative and the Graph of f.

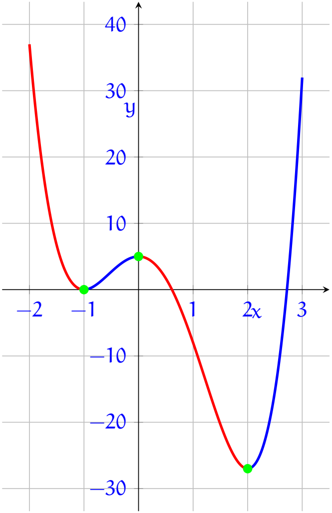
$f'(x) > 0 \implies f(x)$ is increasing.
$f'(x) < 0 \implies f(x)$ is decreasing.

Example (Stewart, p. 287)

Find the intervals where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing or decreasing.

Soln. $f'(x) = 12x^3 - 12x^2 - 24x$.
So $f'(x) = 12x(x+1)(x-2)$.
Critical Points: $x = -1, 0, 2$.

	$f'(x)$	$f(x)$
$x < -1$	< 0	$\searrow$
$-1 < x < 0$	> 0	$\nearrow$
$0 < x < 2$	< 0	$\searrow$
$x > 2$	> 0	$\nearrow$



The First Derivative Test.

The previous example illustrates that:
If f has a local minimum at $x = p$ then f is **decreasing** ($\searrow$) to the left of p and **increasing** ($\nearrow$) to the right of p.
This means that $f'(x)$ **changes its sign** from negative to positive at a local minimum.

First Derivative Test

Suppose that p is a critical point of the continuous function $f(x)$.

1. If $f'(x)$ changes its sign from **negative to positive** at p then $f(x)$ has a **local minimum** at $x = p$.
2. If $f'(x)$ changes its sign from **positive to negative** at p then $f(x)$ has a **local maximum** at $x = p$.
3. If $f'(x)$ does **not change its sign** at p then f has neither a local minimum or a local maximum at $x = p$.

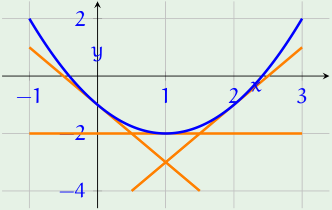
In other words, $f(x)$ has a local extremum at $x = p$ if and only if the derivative $f'(x)$ changes its sign at p.

Concave Up and Concave Down.

Concavity

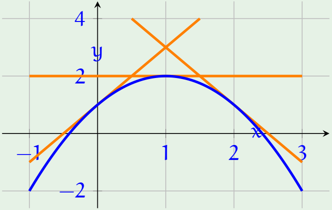
If, for $x \in (a, b)$, the graph of $f(x)$ lies **above all its tangents**, then f is called **concave up** on the interval (a, b).
If, for $x \in (a, b)$, the graph of $f(x)$ lies **below all its tangents**, then f is called **concave down** on the interval (a, b).

$f(x) = x^2 - 2x - 1$



$f(x)$ is **concave up** on the interval $(-1, 3)$.

$f(x) = -x^2 + 2x + 1$



$f(x)$ is **concave down** on the interval $(-1, 3)$.

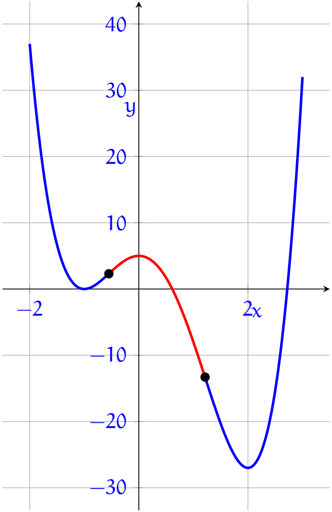
The Graph and the Second Derivative.

The second derivative $f''(x)$ is the slope of the tangent to $f'(x)$.

- If $f''(x) > 0$ for $x \in (a, b)$ then $f(x)$ is concave up on the interval (a, b).
- If $f''(x) < 0$ for $x \in (a, b)$ then $f(x)$ is concave down on the interval (a, b).
- A point p is called an **inflection point** of $f(x)$ if $f''(x)$ changes its sign at $x = p$.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$f''(x) = 12(3x^2 - 2x - 2)$.
Solve $f''(x) = 0$. $\rightsquigarrow$
Inflection points: $x = \frac{1}{3}(1 \pm \sqrt{7})$.



The Second Derivative Test.

We can combine the above facts to obtain another way of testing whether a critical point corresponds to a local minimum or to a local maximum.

Second Derivative Test.

Suppose that a function $f(x)$ is continuous near a point $x = p$, and that $f'(p) = 0$.

1. If, moreover, $f''(p) > 0$ then f has a local minimum at p.
2. If, moreover, $f''(p) < 0$ then f has a local maximum at p.

Example

Recall that $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ has critical points at $x = -1$, 0, and 2. Do these correspond to local maxima or minima?
$f''(x) = 12(3x^2 - 2x - 2)$.
$f''(-1) = 36 > 0$: min, $f''(0) = -24 < 0$: max, $f''(2) = 72 > 0$: min.

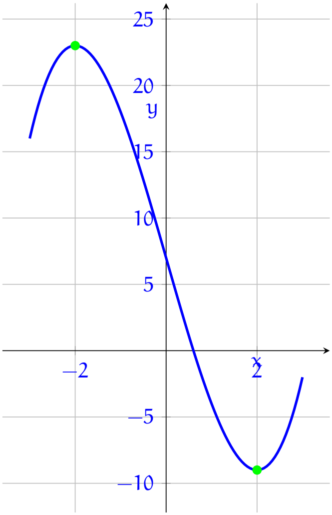
Example.

Example

Find all the critical points of the function $f(x) = x^3 - 12x + 7$. For each critical point, apply the Second Derivative Test to determine whether the point is a local minimum or a local maximum.

$f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$.
$\rightsquigarrow$ critical points: $x = -2, 2$.

$f''(x) = 6x$. 2nd derivative test:
$f''(-2) = -12 < 0$: maximum
$f''(2) = 12 > 0$: minimum.



Graph Sketching.

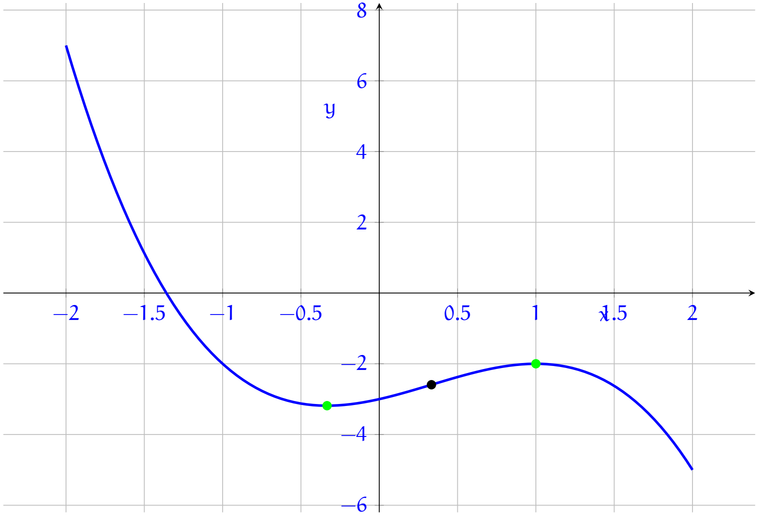
The first and second derivative of a function exhibit properties of the graph of the function and thus can help us to sketch the graph.

MA161 Exam 2013/14 Semester 1

Consider the function $f(x) = -x^3 + x^2 + x - 3$.

1. Locate the critical points of $f(x)$.
2. For each of these, determine if it corresponds to a local maximum or a local minimum.
3. Find the regions where $f(x)$ is increasing, and the regions where $f(x)$ is decreasing.
4. Give a sketch of the graph of f over the interval $[-2, 2]$, showing clearly the regions where it is increasing, where it is decreasing, and where it intersects the y-axis.

The Sketch of the Graph.



Course Review.

Over the past 12 weeks, we have covered the following topics.

1. Sets ($\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, ...). Inequalities.
2. Functions. Even and Odd Functions. Domain and Range.
3. Polynomials. Power Functions. Rational Functions. Trigonometric Functions. Exponentials. Logarithms. Piecewise Defined Functions.
4. Limits. One-Sided Limits. Infinite Limits.
5. Continuity. Limits at Infinity.
6. Tangents and Derivatives. Differentiation from First Principles.
7. Derivatives of Polynomials and Exponentials.
8. Differentiation Rules: Product, Quotient and Chain Rules.
9. Derivatives of Trigonometric Functions. Implicit Differentiation. Derivatives of Inverse Functions and Logarithms.
10. Applications: L'Hôpital's Rule; Critical Points; Local and Global Maxima and Minima; Closed Interval Method; First Derivative Test; Second Derivative Test; Graph Sketching.

MA161/MA1161 Exam Paper 1.

- There are 4 questions on the paper, 2 on Algebra and 2 on Calculus.
- You should **answer all questions**. All questions carry the same marks.
- This exam accounts for 30% of the overall grade. The Semester 2 exam also accounts for 30%. The Continuous Assessment (online, semester 1 and 2) is worth 40%.

To complete MA161:

- you need at least 35% of the Continuous Assessment.

To pass this module:

- you need at least 40% in your overall grade.

This means:

- You can still pass the course if you get less that 40% in Paper 1.
- However, if you repeat the exam you will have to **repeat both papers**.
- The Continuous Assessment cannot be repeated.

Revision.

To prepare for the exam:

- Study the online notes and worked examples from the lectures.
- There are copies of the course textbook in the library.
- Review your online homework assignments (solutions are also online).
- Do the **two** sets of revision exercises at the end of this document.
- Look at some previous exam papers.

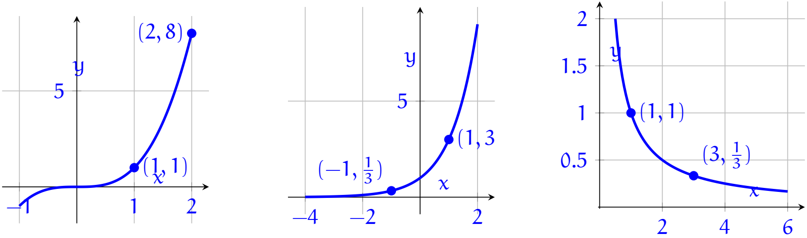
Outline of Calculus Questions.

The 2 Calculus questions on the paper are organized more or less in the order we covered them in class.

1. **Functions and Limits:** Catalogue of Functions (Polynomials, Power Functions, Exponential Functions, . . .); Even and Odd Functions; Inequalities; Domain and Range of a Function; Two-Sided Limits; One-Sided Limits; Limits at Infinity; Continuity; Derivatives as Limits (First Principles).
2. **Differentiation:** Product, Quotient and Chain Rules; Derivatives of Trigonometric, Logarithmic and Exponential Functions; Critical Points; Maxima and Minima; Graph Sketching.

Revision Exercises Q1.

(a) The three graphs below are of the functions $P(x) = a^x$, $Q(x) = x^b$ and $R(x) = x^{-c}$, where a, b and c are positive integers. Determine which graph corresponds to which function, and find a, b and c.



(b) Solve the following inequalities:

(i) $18 - 2x^2 \leqslant 0$. (ii) $1 + 2|x - 3| > 7$.

Revision Exercises Q1.

(c) According to Wikipedia, the isotope radon-222 has a half-life of 4 days, which means that, compared to time t (days), only half of the substance remains at time $t+4$.

(i) If there are 100g present at time $t=0$, show that the quantity remaining at time t can be described by the function

$$f(t) = 100 \cdot (\tfrac{1}{2})^{t/4}.$$

(ii) How many days must pass for the mass to reduce from 100g to less than 1g?

(d) For each of the following functions, determine if it is even, odd, or neither.

(i) $f(x) = -1 - x^2 - x^4$.

(ii) $f(x) = e^x - e^{-x}$.

(e) Evaluate the following limits:

(i) $\displaystyle\lim_{x\to-8} \frac{x^2+11x+24}{x+8}$;

(ii) $\displaystyle\lim_{x\to\infty} \frac{x^4-4x^2+4}{x^5+4x^3}$.

Revision Exercises Q2.

(a) (i) Determine if the function $f(x) = \frac{x^3}{|x|}$ is continuous at $x = 0$.

(ii) Determine if the function

$$f(x) = \begin{cases} 4x^2 - 2x + 1, & x \leqslant -1, \\ 6 - x, & -1 < x \leqslant 1, \\ 1 + (1+x)^2, & x > 1, \end{cases}$$

is continuous for all $x \in \mathbb{R}$.

(b) Use that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to show that the derivative of $f(x) = 2\sqrt{x}$ is $f'(x) = \frac{1}{\sqrt{x}}$.

(c) Find the derivatives of the following functions:

(i) $f(x) = \ln(x^2 + 1)$;

(ii) $f(x) = \frac{\cos(x^3 + 2)}{x + 1}$.

Revision Exercises Q2.

(d) Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

(e) A chemist models the temperature of a reacting mixture in an experiment as

$$f(t) = 10 + 5t - \ln(1 + 40t),$$

where t is time in minutes, and temperature is measured in degrees Celsius. The experiment runs for one minute: from $t = 0$ to $t = 1$.

(i) Find the maximum and minimum temperature of the mixture during the experiment.

(ii) Give a sketch of the graph of f, showing clearly the regions where the temperature is increasing, and where it is decreasing.