Week 11: Applications of Differentiation.

MA161/MA1161: Semester 1 Calculus.

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Recall: Differentiation.

We have seen that a function $f(x)$ can have a **derivative**, written as $f'(x)$, or as $\frac{d}{dx}f(x)$, and defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- Geometrically, $f'(x)$ is the **slope of the tangent** to f at x.
- The derivative of $f(x)$ it tells how $f(x)$ is changing at x.
- Derivatives can be computed **from first principles**, that is directly from the definition.
- Usually, it is easier to calculate derivatives with the help of various **differentiation rules**, that have been derived from the definition.

Differentiation Rules.

1. $\big(C\,f(x)\big)' = C\,f'(x)$, for any constant $C \in \mathbb{R}$.
2. $\big((f(x) + g(x)\big)' = f'(x) + g'(x)$.
3. **Power Rule**: $\big(x^a\big)' = a x^{a-1}$, for $a \in \mathbb{R}$.
4. **Exponential function**: $\big(e^x\big)' = e^x$.
5. **Logarithm**: $\big(\ln(x)\big)' = \dfrac{1}{x}$.
6. **Trigonometric functions**:
   $\big(\sin(x)\big)' = \cos(x)$, $\big(\cos(x)\big)' = -\sin(x)$, $\big(\tan(x)\big)' = \dfrac{1}{\cos^2(x)}$.

7. **Product Rule**: $\big(u(x)\,v(x)\big)' = u'(x)\,v(x) + u(x)\,v'(x)$.
8. **Quotient Rule**: $\left(\dfrac{u(x)}{v(x)}\right)' = \dfrac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)^2}$.
9. **Chain Rule**: $\big(u(v(x))\big)' = u'(v(x))\,v'(x)$ (where $u' = \frac{d}{dv}u$.)

Differentiating Logarithmic Functions.

Differentiate $f(x) = \ln(5 + 2x)$.

Write $f(x) = u(v(x))$ and apply the Chain Rule.
$u(v) = \ln v \implies u'(v) = \frac{1}{v}$.
$v(x) = 5 + 2x \implies v'(x) = 2$.
Chain Rule: $f'(x) = u'(v) \cdot v'(x) = \dfrac{2}{v} = \dfrac{2}{5 + 2x}$.

Differentiate $f(x) = \ln(x^2 + 1)$ (MA161 Exam 2012/13).

Write $f(x) = u(v(x))$ and apply the Chain Rule.
$u(v) = \ln v \implies u'(v) = \frac{1}{v}$.
$v(x) = x^2 + 1 \implies v'(x) = 2x$.
Chain Rule: $f'(x) = u'(v) \cdot v'(x) = \dfrac{2x}{v} = \dfrac{2x}{x^2 + 1}$.

Indeterminate Forms.

Remember (from weeks 5, 6, or 7) **limits** like

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

The limit law for quotients, $\lim_{x\to p} \big(f(x)/g(x)\big) = \big(\lim_{x\to p} f(x)\big)/\big(\lim_{x\to p} g(x)\big)$, does not apply here, since $\lim_{x\to p} g(x) = 0$.

We solved the above limit by factorizing $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$, for $x \neq 1$, and then computing the limit of $x+1$ as $x \to 1$.
Such a strategy works, if both f and g are polynomials, but not for limits like

$$\lim_{x\to 0} \frac{\sin(x)}{x} \quad \text{or} \quad \lim_{x\to 1} \frac{\ln(x)}{x-1}.$$

However, if **both** $f(p) = 0$ **and** $g(p) = 0$ then the limit $\lim_{x\to p} \big(f(x)/g(x)\big)$ is called an **indeterminate form**, sometimes written as $\left[\frac{0}{0}\right]$.
It turns out that **differentiation** can be used to try and solve such indeterminate forms.

L'Hôpital's Rule.

L'Hôpital's Rule.

Suppose that f and g are differentiable functions, and that $g(x) \neq 0$ on an interval near (but not at) $x = p$.
If both $f(x) \to 0$ and $g(x) \to 0$ as $x \to p$, or if both $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to p$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)}.$$

That is, instead of computing the limit of the quotient of $f(x)$ and $g(x)$, we compute the limit of the quotient of their **derivatives**.

Examples

$$\lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \frac{1/x}{1} = 1. \qquad \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1.$$

Computing Indeterminate Forms.

Example (MA161 Paper 1, 2012/13)

Evaluate $\lim\limits_{x \to -8} \dfrac{x^2 + 11x + 24}{x + 8}$.

(Done already in week 6, but in a different way.)

Solution. Set $f(x) = x^2 + 11x + 24$ and $g(x) = x + 8$.

Compute the limits:
$f(x) \to f(-8) = 64 - 88 + 24 = 0$ as $x \to -8$.
$g(x) \to g(-8) = 0$ as $x \to -8$.
So $\lim\limits_{x \to -8} \dfrac{x^2 + 11x + 24}{x + 8} = \left[\dfrac{0}{0}\right]$ is an indeterminate form.

Find the derivatives:
$f'(x) = 2x + 11$, $g'(x) = 1$.

Apply l'Hôpital's Rule:
$$\lim_{x \to -8} \frac{f(x)}{g(x)} = \lim_{x \to -8} \frac{f'(x)}{g'(x)} = \lim_{x \to -8} \frac{2x + 11}{1} = -16 + 11 = -5.$$

Optimization.

Typical **Optimization Problems**:

- What is the **fastest** way to get from A to B?
- What is the **cheapest** way to get from A to B?
- What type of packaging encloses the **largest volume**?
- At what speed does this car use the **least amount** of gallons per mile?
- ...
- For which value of x does the function $f(x)$ yield the **largest value**?

If an optimization problem can be modelled with a function f of a real variable x, then plotting the graph of $f(x)$ may exhibit regions of extremal behavior of $f(x)$.
The derivative of $f(x)$ can help to identify those points x for which the outcome $f(x)$ is optimal (maximal/minimal).

Absolute Maximum and Minimum.

The function $f(x)$ has an **absolute maximum** at the point p if

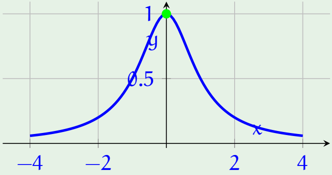
$$f(p) \geqslant f(x) \text{ for all points } x \text{ in the domain of } f.$$

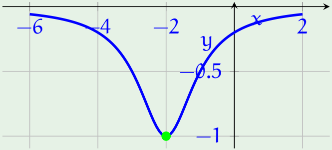
We then call $f(p)$ the **maximum value** of $f(x)$.

The function $f(x)$ has an **absolute minimum** at the point p if

$$f(p) \leqslant f(x) \text{ for all points } x \text{ in the domain of } f.$$

We then call $f(p)$ the **minimum value** of $f(x)$.

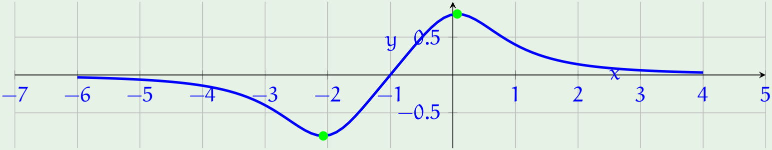




Extreme Values.

The minimum and maximum values of $f(x)$ are called the **extreme** values.

Consider the function $f(x) = \dfrac{1}{x^2+1} - \dfrac{1}{(x+2)^2+1}$.



$f(x)$ has **extreme points** at $x = -1 + \dfrac{\sqrt{2}}{\sqrt[4]{3}}$ and at $x = -1 - \dfrac{\sqrt{2}}{\sqrt[4]{3}}$,

where $\sqrt{2}/\sqrt[4]{3} \approx 1.07457$.

These points correspond to the **extreme values**

$f(-1-\sqrt{2}/\sqrt[4]{3}) = -\dfrac{3^{3/4}}{2\sqrt{2}}$ and $f(-1+\sqrt{2}/\sqrt[4]{3}) = \dfrac{3^{3/4}}{2\sqrt{2}} \approx 0.80593$.

Local Maximum and Minimum.

A function $f(x)$ may have a point p where the function value $f(p)$ is **larger than any value nearby**, but p is **not an absolute maximum**.

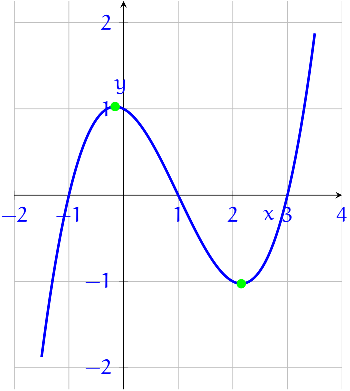
The function $f(x)$ has a **local maximum** at the point p if

$$f(p) \geqslant f(x) \text{ for all } x \text{ near } p.$$

The function $f(x)$ has a **local minimum** at the point p if

$$f(p) \leqslant f(x) \text{ for all } x \text{ near } p.$$

By "all x near p" we mean all x in an open interval that contains p.



Fermat's Theorem.

The following theorem gives a **necessary condition** for $x = p$ to be a local extremum of a function $f(x)$, in terms of the derivative of f.

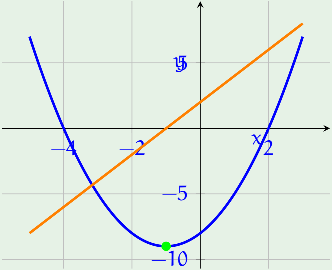
Fermat's Theorem

If $f(x)$ has a local maximum or minimum at the point $x = p$, and if $f'(p)$ exists, then $f'(p) = 0$.

Geometrically, if $f(x)$ has a **local extremum** at p then the tangent to $f(x)$ at $x = p$ is parallel to the x-axis and so has **slope zero**.

The function $f(x) = x^2 + 2x - 8$ has a local minimum at $x = -1$.

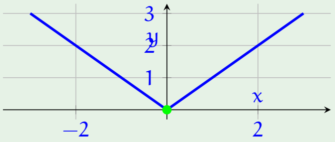
Check the derivative:
$f'(x) = 2x + 2$.
$f'(-1) = -2 + 2 = 0$,
as predicted by the theorem.



Logic.

It is important to understand the logic of Fermat's Theorem.

$f(x) = |x|$ has a local (even a global) minimum at $x = 0$, but $f'(x)$ does not exist.



For a differentiable function $f(x)$, the theorem tells us:

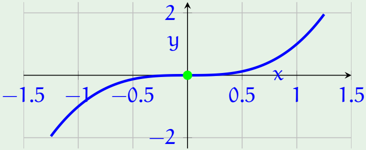
If $f(x)$ has a local extremum at $x = p$ then $f'(p) = 0$.

This means in turn:

If $f'(p) \neq 0$ then $f(x)$ does **not** have an extremum at $x = p$.

But this does not necessarily mean that if $f'(p) = 0$ then f has a local extremum at $x = p$.

$f(x) = x^3$ has $f'(x) = 3x^2$ and $f'(0) = 0$, but $x = 0$ is not a local extremum of $f(x)$.



Critical Points.

The point $x = p$ is a **critical point** of the function $f(x)$
if $f'(p)$ does not exist, or if $f'(p) = 0$.

To locate the local minima and maxima of $f(x)$:

1. Find all critical points of $f(x)$.
2. Check how $f(x)$ behaves near these points.

Example (MA161 Semester 1 Exam 2013/14)

Find all the critical points $f(x) = -x^3 + x^2 + x - 3$.

Solution. $f'(x) = -3x^2 + 2x + 1$ exists for all x:
the only critical points are those x where $f'(x) = 0$.

$$f'(x) = -3x^2 + 2x + 1 = -3(x^3 + \tfrac{2}{3}x + \tfrac{1}{3}) = -3(x-1)(x+\tfrac{1}{3}) = 0,$$
$$x = 1 \text{ or } x = -\tfrac{1}{3}.$$

The critical points are $x = 1$ and $x = -\frac{1}{3}$.

Locating Minima and Maxima.

The Closed Interval Method.

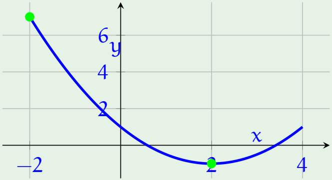
To find the **absolute** maximum or minimum value of a **continuous** function $f(x)$ on a **closed interval** $[a, b]$:

1. Find the critical points of $f(x)$ in (a, b) and, for each critical point p, determine the value $f(p)$.
2. Find the values $f(x)$ for $x = a$ and $x = b$.
3. The largest of the values found in steps 1–2 is the **absolute maximum**, the smallest is the **absolute minimum** of f on $[a, b]$.

Example ($f(x) = \frac{1}{2}x^2 - 2x + 1$)

Find the absolute maximum and minimum of $f(x)$ on $[-2, 4]$.

1. $f'(x) = x - 2 = 0 \Leftrightarrow x = 2$ and $f(2) = -1$.
2. $f(-2) = 7$, $f(4) = 1$.
3. Absolute maximum: $f(-2) = 7$. Absolute minimum: $f(2) = -1$.



Extreme Temperatures.

Example

A chemist models the **temperature of a reacting mixture** in an experiment as $f(t) = 10 + 5t - \ln(1 + 40t)$, where t is time in minutes, and the temperature is measured in degrees Celsius. The experiment runs for one minute: from $t = 0$ to $t = 1$.
Find the **maximum** and the **minimum temperature** of the mixture during the experiment.

Solution. Apply the Closed Interval Method.

1. $f'(t) = 0 + 5 - \frac{40}{1+40t}$ (Chain Rule)
$f'(t) = 0 \Leftrightarrow 5 = \frac{40}{1+40t} \Leftrightarrow t = \frac{7}{40}$. And $f(\frac{7}{40}) \approx 8.796$.

2. $a = 0$, $b = 1$. $f(0) = 10$ and $f(1) = 15 - \ln(41) \approx 11.286$.

3. The absolute maximum of $f(t)$ on $[0, 1]$ is $f(1) \approx 11.286$. The absolute minimum of $f(t)$on $[0, 1]$ is $f(\frac{7}{40}) \approx 8.796$.

The Derivative and the Graph of f.

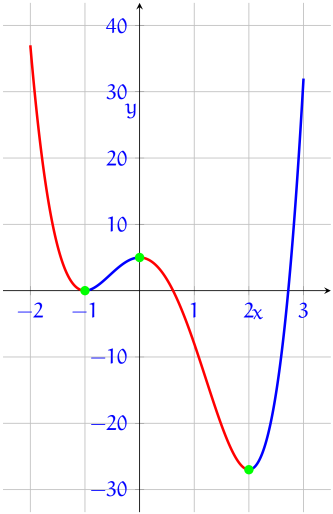
$f'(x) > 0 \implies f(x)$ is increasing.
$f'(x) < 0 \implies f(x)$ is decreasing.

Example (Stewart, p. 287)

Find the intervals where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing or decreasing.

Soln. $f'(x) = 12x^3 - 12x^2 - 24x$.
So $f'(x) = 12x(x+1)(x-2)$.
Critical Points: $x = -1, 0, 2$.

	$f'(x)$	$f(x)$
$x < -1$	< 0	$\searrow$
$-1 < x < 0$	> 0	$\nearrow$
$0 < x < 2$	< 0	$\searrow$
$x > 2$	> 0	$\nearrow$



The First Derivative Test.

The previous example illustrates that:
If f has a local minimum at $x = p$ then f is **decreasing** ($\searrow$) to the left of p and **increasing** ($\nearrow$) to the right of p.
This means that $f'(x)$ changes its sign from negative to positive at a local minimum.

First Derivative Test

Suppose that p is a critical point of the continuous function $f(x)$.

1. If $f'(x)$ changes its sign from **negative to positive** at p then $f(x)$ has a **local minimum** at $x = p$.
2. If $f'(x)$ changes its sign from **positive to negative** at p then $f(x)$ has a **local maximum** at $x = p$.
3. If $f'(x)$ does **not change its sign** at p then f has neither a local minimum or a local maximum at $x = p$.

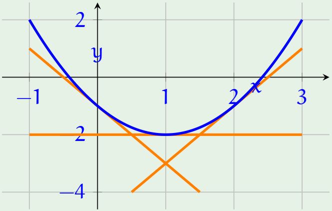
In other words, $f(x)$ has a local extremum at $x = p$ if and only if the derivative $f'(x)$ changes its sign at p.

Concave Up and Concave Down.

Concavity

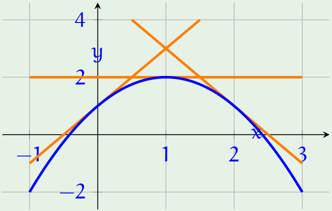
If, for $x \in (a, b)$, the graph of $f(x)$ lies **above all its tangents**, then f is called **concave up** on the interval (a, b).
If, for $x \in (a, b)$, the graph of $f(x)$ lies **below all its tangents**, then f is called **concave down** on the interval (a, b).

$f(x) = x^2 - 2x - 1$



$f(x)$ is **concave up** on the interval $(-1, 3)$.

$f(x) = -x^2 + 2x + 1$

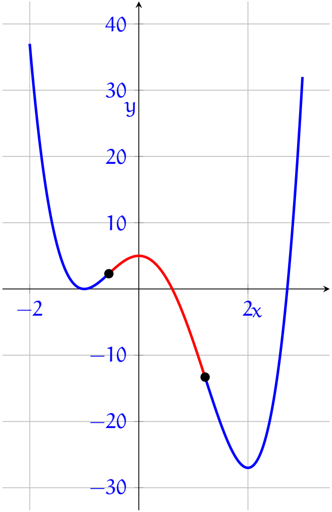


$f(x)$ is **concave down** on the interval $(-1, 3)$.

The Graph and the Second Derivative.

The second derivative $f''(x)$ is the slope of the tangent to $f'(x)$.

- If $f''(x) > 0$ for $x \in (a, b)$ then $f(x)$ is concave up on the interval (a, b).
- If $f''(x) < 0$ for $x \in (a, b)$ then $f(x)$ is concave down on the interval (a, b).
- A point p is called an **inflection point** of $f(x)$ if $f''(x)$ changes its sign at $x = p$.



$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$f''(x) = 12(3x^2 - 2x - 2)$.
Solve $f''(x) = 0$. $\rightsquigarrow$
Inflection points: $x = \frac{1}{3}(1 \pm \sqrt{7})$.

Exercises.

1. Use, if possible, l'Hôpital's Rule to evaluate the following limits.

   (a) $\lim_{x\to 2} \frac{2x^2-1}{x-1}$. (b) $\lim_{x\to -1} \frac{x^3+1}{x+1}$. (c) $\lim_{x\to \pi/2} \frac{\cos x}{x-\pi/2}$.

2. Find the critical points of the following functions.
   (a) $f(x) = 2x^3 - 3x^2 - 36x$.
   (b) $g(t) = t^4 + t^3 + t^2 + 1$.
   (c) $f(x) = |3x - 4|$.
   (d) $g(x) = 2\cos x + \sin^2 x$.
   (e) $f(x) = x^2 e^{-3x}$.

3. Find the absolute maximum and the absolute minimum of $f(x)$ in the given interval.
   (a) $f(x) = 12 + 4x - x^2$, $x \in [0, 5]$.
   (b) $f(x) = 3x^4 + 4x^3 + 12x^2 + 1$, $x \in [-2, 3]$.
   (c) $f(x) = (x^2 - 1)^3$, $x \in [-1, 2]$.
   (d) $f(x) = \frac{x}{x^2 - x + 1}$, $x \in [0, 3]$.
   (e) $f(x) = x - \ln x$, $x \in [\frac{1}{2}, 2]$.
   (f) $f(x) = \ln(x^2 + x + 1)$, $x \in [-1, 1]$.

Exercises.

4. Show that $p = 5$ is a critical point of the function

$$g(x) = 2 + (x - 5)^3,$$

but that $g(x)$ does not have a local extremum at $x = 5$.

5. Show that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

6. For each function f, (i) find the intervals on which f is increasing or decreasing, (ii) find the local minimum and maximum values, (iii) find the intervals of concavity and the inflection points. (iv) Use the information from (i)–(iii) to sketch the graph of f.
   (a) $f(x) = x^3 - 12x + 2$.
   (b) $f(x) = (x + 1)^5 - 5x - 2$.
   (c) $f(x) = 5x^{2/3} - 2x^{5/3}$.
   (d) $f(x) = x - \sin x$, $x \in [0, 4\pi]$.
   (e) $f(x) = \ln(x^4 + 27)$.