Week 10: Differentiation Techniques.

MA161/MA1161: Semester 1 Calculus.

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Recall: Differentiation.

We have seen that a function $f(x)$ has a **derivative**, written as $f'(x)$, or as $\frac{d}{dx}f(x)$, and defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Geometrically, $f'(x)$ is the **slope of the tangent** to f at x: it tells how $f(x)$ is changing at x.
Derivatives can be computed **from first principles**, that is directly from the definition.
From the definition, one can derive various **differentiation rules**, that make the calculation of derivatives relatively easy.

Differentiation Rules.

1. $\big(C\,f(x)\big)' = C\,f'(x)$, for any constant $C \in \mathbb{R}$.
2. $\big((f(x) + g(x)\big)' = f'(x) + g'(x)$.
3. **Power Rule**: $\big(x^a\big)' = a x^{a-1}$, for $a \in \mathbb{R}$.
4. **Exponential function**: $\big(e^x\big)' = e^x$.
5. **Product Rule**: $\big(u(x)\,v(x)\big)' = u'(x)\,v(x) + u(x)\,v'(x)$.
6. **Quotient Rule**: $\left(\dfrac{u(x)}{v(x)}\right)' = \dfrac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)^2}$.
7. **Trigonometric functions**:
   $\big(\sin(x)\big)' = \cos(x)$, $\big(\cos(x)\big)' = -\sin(x)$, $\big(\tan(x)\big)' = \dfrac{1}{\cos^2(x)}$.
8. **Chain Rule**: $\big(u(v(x))\big)' = u'(v(x))\,v'(x)$ (where $u' = \frac{d}{dv}u$.)

The Chain Rule.

In order to compute the derivative of the **composite function** $(u \circ v)(x) = u(v(x))$, one regards u **as a function of** v (and v as a function of x).

Then one determines the derivative $\frac{d}{dv}u$ of u (wrt. v) and the derivative $\frac{d}{dx}$ of v (wrt. x).

The derivative of $u(v(x))$ (wrt. x) then is the **product** of those two.

Chain Rule

If u and v are functions so that v is differentiable at x and u is differentiable at $v(x)$ then $u \circ v$ is differentiable at x and its derivative (wrt. x) is

$$(u \circ v)'(x) = u'(v(x)) \cdot v'(x).$$

The chain rule is sometimes written as

$$\frac{d}{dx}u = \frac{d}{dv}u \cdot \frac{d}{dx}v; \quad \text{or even as} \quad \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}.$$

The Chain Rule: Examples.

Differentiate $\sin(x^2)$.

Set $u(v) = \sin(v)$ and $v(x) = x^2$, then $\sin(x^2) = u(v(x))$.
We have $\frac{d}{dv}u(v) = (\sin(v))' = \cos(v)$ and $\frac{d}{dx}v(x) = (x^2)' = 2x$.
Hence $(\sin(x^2))' = u'(v(x)) \cdot v'(x) = \cos(x^2) \cdot 2x$.

Examples

- Find $f'(x)$ when $f(x) = \sqrt{x^2 + 1}$. (Source: Stewart, p. 198)
- Differentiate $f(x) = (\sin(2x))^6$. (Apply the Chain Rule repeatedly.)
- Differentiate $f(x) = \dfrac{\cos(x^3 + 2)}{x + 1}$. (MA161 Paper 1, 2012/13)
- Suppose that $A(r)$ represents the area of a circle of radius r, i.e., $A(r) = \pi r^2$. The radius in turn can be expressed in terms of the circumference ℓ of the circle as $r(\ell) = \frac{\ell}{2\pi}$. How does the area change with respect to the circumference?

Tangents.

Now that we know how to compute the derivative of a function $f(x)$, we can use this to find the **formula for the tangent** to f at a point.

Idea

The equation for the line through the point (x_0, y_0) with slope m is

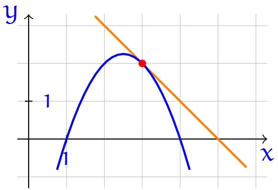
$$y = m(x - x_0) + y_0.$$

The tangent to f at x_0 is the line through the point $(x_0, f(x_0))$ with slope $m = f'(x_0)$. Hence its equation is

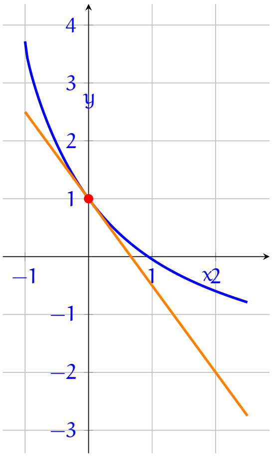
$$y = f'(x)\,(x - x_0) + f(x_0).$$

Example

What is the equation of the tangent to $f(x) = -x^2 + 5x - 4$ at $x = 3$?



Tangents: Example.



Find the equation for the tangent to

$$f(x) = e^{-x} - \sqrt{x+1} + 1$$

at $x = 0$.

Solution. From

$$f'(x) = -e^{-x} - \frac{1}{2\sqrt{x+1}},$$

we get $f(0) = 1$ and $f'(0) = -\frac{3}{2}$, whence

$$y = 1 - \tfrac{3}{2}x$$

is the equation of the tangent.

Implicitly Defined Functions.

So far, all our functions have been given by an **explicit formula**, like $f(x) = 2x + e^{-x}$, or $f(x) = \sin(x)/x$.
There are, however, many useful equations, that **implicitly** define a function, for example

$$x^2 + y^2 = 5^2,$$

the equation of a circle of radius 5 centered at the origin $(0, 0)$.
Often it is possible to differentiate an implicitly defined function, without first finding an explicit formula.
In this way, we can for example determine the slope of the tangent to the graph of an equation at a particular point.
For this:

1. Differentiate both sides of the equation,
2. Solve for the derivative.

This process requires careful use of the Chain Rule.

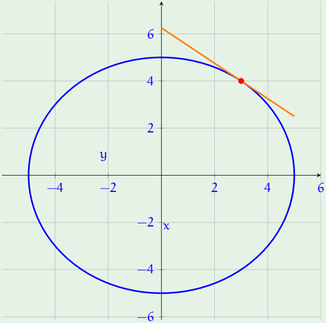
Implicit Differentiation.

Example (MA161 Paper 1, 2012/13)

Find the equation of the tangent to the circle

$$x^2 + y^2 = 25$$

at the point $(3, 4)$.



Solution. 1. Differentiate both sides:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}25$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$
$$2x + \frac{d}{dy}(y^2) \cdot \frac{d}{dx}y = 0$$
$$2x + 2y \cdot \frac{d}{dx}y = 0$$

2. Solve for $\frac{d}{dx}y$:

$$\frac{d}{dx}y = -x/y.$$

The slope at $(x, y) = (3, 4)$ is $m = -\frac{3}{4}$.
Equation of the tangent:

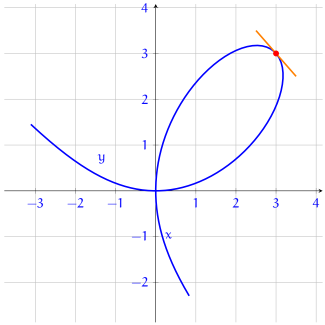
$$y = -\frac{3}{4}(x - 3) + 4.$$

The Folium of Descartes.

The **Folium of Descartes** is a curve defined by an equation like

$$x^3 + y^3 = 6xy,$$

which cannot easily be solved for x or y.



Find the equation of the tangent to the curve at $(3, 3)$.

Soln. Differentiate and solve for $\frac{d}{dx}y$:

$$\begin{aligned}
\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}6xy \\
\frac{d}{dx}(x^3) + \frac{d}{dy}(y^3) \cdot \frac{dy}{dx} &= 6\frac{d}{dx}xy \\
3x^2 + 3y^2 \cdot \frac{dy}{dx} &= 6\left(y + x\frac{dy}{dx}\right) \\
(3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\
\frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}.
\end{aligned}$$

The slope at $(x, y) = (3, 3)$ is $m = -1$.
Equation of the tangent:

$$y = -(x - 3) + 3 = 6 - x$$

Inverse Functions.

Recall: a function g is the **inverse function** of a function f, if $g(f(x)) = x$ and $f(g(x)) = x$ for all x. We then write f^{-1} for g.

Use implicit differentiation to find the derivative of an inverse function.

Example ($\frac{d}{dx}\sin^{-1}(x) = (1 - x^2)^{-1/2}$)

Set $y = \sin^{-1}(x)$. Find $\frac{dy}{dx}$ by differentiating the equation $\sin(y) = x$.

$$\frac{d}{dx}\sin(y) = \frac{d}{dx}x$$
$$\frac{d}{dy}\sin(y) \cdot \frac{dy}{dx} = 1$$
$$\cos(y) \cdot \frac{dy}{dx} = 1.$$

Hence $\frac{dy}{dx} = \frac{1}{\cos(y)}$. Using $\sin^2(y) + \cos^2(y) = 1$, we can write $\cos(y) = \sqrt{1 - x^2}$ whence

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}.$$

The Derivative of the Natural Logarithm.

Recall: The **natural logarithm** $\ln x$ is the inverse function of the **exponential function** e^x. That is:

$$\ln(e^x) = x \text{ and } e^{\ln x} = x.$$

Also: $\frac{d}{dx}e^x = e^x$. It follows that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Proof. Set $y = \ln x$ and find $\frac{dy}{dx}$ by differentiating the equation $e^y = x$.

$$\begin{aligned} \frac{d}{dx}e^y &= \frac{d}{dx}x \\ \frac{d}{dy}e^y \cdot \frac{dy}{dx} &= 1 \\ e^y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= 1/e^y = 1/x. \end{aligned}$$

□

Differentiating Logarithmic Functions.

Differentiate $f(x) = \ln(5 + 2x)$.

Write $f(x) = u(v(x))$ and apply the Chain Rule.
$u(v) = \ln v \implies u'(v) = \frac{1}{v}$.
$v(x) = 5 + 2x \implies v'(x) = 2$.
Chain Rule: $f'(x) = u'(v) \cdot v'(x) = \dfrac{2}{v} = \dfrac{2}{5 + 2x}$.

Differentiate $f(x) = \ln(x^2 + 1)$ (MA161 Exam 2012/13).

Write $f(x) = u(v(x))$ and apply the Chain Rule.
$u(v) = \ln v \implies u'(v) = \frac{1}{v}$.
$v(x) = x^2 + 1 \implies v'(x) = 2x$.
Chain Rule: $f'(x) = u'(v) \cdot v'(x) = \dfrac{2x}{v} = \dfrac{2x}{x^2 + 1}$.

Exercises.

1. Use the Chain Rule to differentiate the following functions.
   (a) $f(x) = \dfrac{\cos(2x)}{x^3}$.
   (b) $f(x) = \ln(\sqrt{x^2 + 1})$.
   (c) $f(x) = \sin(5x^2)$.
   (d) $f(x) = \sin^2(5x)$.
2. Find the tangents to the function $f(x) = x^2 + x - 6$ at the points $x = -4$ and $x = 2$.
3. Use implicit differentiation to show that
   (a) $\dfrac{d}{dx}\cos^{-1}(x) = \dfrac{-1}{\sqrt{1 - x^2}}$.
   (b) $\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1 + x^2}$. [Hint: $\dfrac{d}{dx}\tan(x) = 1 + \tan^2(x)$.]
4. Find the tangents to the curve $x^2 + xy + 2y^2 = 4$ at the points $(1, 1)$ and $(2, -1)$.