Week 9: Differentiation Rules.

MA161/MA1161: Semester 1 Calculus.

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Assignments.

Problem Set 3 is due for submission on Friday before 5pm.

Remember: MA161 students need to score at least 35% in these assignments in order to complete the module.

That's an average of 5.25 points (out of 15) on each of the (four) problem sets.

Assignments cannot and will not be repeated.
Need help?

- Go to SUMS.
- Go to tutorials.
- Ask questions at lectures.

Recall: The Derivative.

Geometrically, the **derivative** of a function f at a point p is the **slope of the tangent** to f at p.
The formal, mathematical definition is:

The derivative of a function $f\colon D \to \mathbb{R}$ at a point p is defined as

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h},$$

provided the limit exists.

The derivative of a function f at a point p is the instantaneous **rate of change** of f at p.
Derivatives can be computed **from first principles**, that is directly from the above definition.
Differentiation rules are formulas which simplify differentiation tasks, and apply to wide ranges of functions.

The Product Rule.

[Section 3.2 of the Book]

The derivative of a product uv of functions u and v is **not** the product of the derivatives of u and v.

Product Rule

If u and v are differentiable functions then

$$(u\,v)' = u'\,v + u\,v'.$$

Differentiate $f(x) = x^2\,e^x$.

Solution. Read $f(x)$ as $u(x)\,v(x)$, where

$$u(x) = x^2 \textit{ and } v(x) = e^x.$$

Then

$$u'(x) = 2x \textit{ and } v'(x) = e^x.$$

Hence

$$f'(x) = u(x)\,v'(x) + u'(x)\,v(x) = x^2\,e^x + 2x\,e^x = (x^2 + 2x)e^x.$$

Applications of the Product Rule.

Use the product rule to differentiate $f(x) = x^3$.

$f(x) = x^3 = x \cdot x^2 = u(x)\, v(x)$, where $u(x) = x$ and $v(x) = x^2$.
Then $u'(x) = 1$ and $v'(x) = 2x$, and

$$f'(x) = u(x)\, v'(x) + u'(x)\, v(x) = x \cdot 2x + 1 \cdot x^2 = 3x^2.$$

Differentiate $f(x) = (2x^3 + 1)(3x - 2)$.

$u(x) = 2x^3 + 1 \implies u'(x) = 6x^2$ and $v(x) = 3x - 2 \implies v'(x) = 3$.

$$f'(x) = 3(2x^3 + 1) + 6x^2(3x - 2) = 24x^3 - 12x^2 + 3.$$

Example (MA161 Semester 1 Exam 2013/14)

Find the derivative of $f(x) = 4x^3 - x^2 e^x$.

$$\begin{aligned} f'(x) &= (4x^3)' - (x^2 e^x)' = 12x^2 - ((x^2)(e^x)' + (x^2)'(e^x)) \\ &= 12x^2 - (x^2 e^x + 2xe^x) = 12x^2 - (x^2 + 2x)e^x. \end{aligned}$$

The Quotient Rule.

Quotient Rule

If u and v are differentiable functions, and if $v \neq 0$ then

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

Differentiate $f(x) = e^x/x$.

Solution. Read $f(x)$ as $u(x)/v(x)$, where $u(x) = e^x$ and $v(x) = x$. Then $u'(x) = e^x$ and $v'(x) = 1$. Hence

$$f'(x) = (u'(x)\,v(x) - u(x)\,v'(x))/v(x)^2 = (e^x x - e^x)/x^2.$$

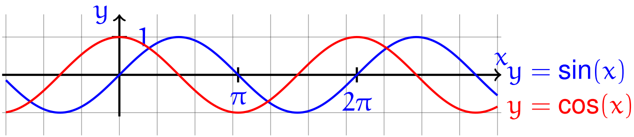
Differentiate $g(x) = (1 + x^2)/(1 + x)$.

$u(x) = 1 + x^2 \implies u'(x) = 2x$ and $v(x) = 1 + x \implies v'(x) = 1$.

$$f'(x) = (2x(1+x) - (1+x^2))/(1+x)^2 = (x^2+2x-1)/(1+x)^2.$$

Derivatives of Trigonometric Functions.

Recall: The graphs of $\sin(x)$ and $\cos(x)$ look like:



Given that the derivative of $f(x)$ at $x = p$ is the slope of the tangent to $f(x)$ at p, it is not hard to believe that:

$\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$.

Then, using the Quotient Rule, one finds that:

$$(\tan(x))' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

The Chain Rule.

In order to compute the derivative of the **composite function** $(u \circ v)(x) = u(v(x))$, one regards u **as a function of** v (and v as a function of x).

Then one determines the derivative $\frac{d}{dv}u$ of u (wrt. v) and the derivative $\frac{d}{dx}$ of v (wrt. x).

The derivative of $u(v(x))$ (wrt. x) then is the **product** of those two.

Chain Rule

If u and v are functions so that v is differentiable at x and u is differentiable at $v(x)$ then $u \circ v$ is differentiable at x and its derivative (wrt. x) is

$$(u \circ v)'(x) = u'(v(x)) \cdot v'(x).$$

The chain rule is sometimes written as

$$\frac{d}{dx}u = \frac{d}{dv}u \cdot \frac{d}{dx}v; \quad \text{or even as} \quad \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}.$$

The Chain Rule: Examples.

Differentiate $\sin(x^2)$.

Set $u(v) = \sin(v)$ and $v(x) = x^2$, then $\sin(x^2) = u(v(x))$.
We have $\frac{d}{dv}u(v) = (\sin(v))' = \cos(v)$ and $\frac{d}{dx}v(x) = (x^2)' = 2x$.
Hence $(\sin(x^2))' = u'(v(x)) \cdot v'(x) = \cos(x^2) \cdot 2x$.

Examples

- Find $f'(x)$ when $f(x) = \sqrt{x^2 + 1}$. (Source: Stewart, p. 198)
- Differentiate $f(x) = (\sin(2x))^6$. (Apply the Chain Rule repeatedly.)
- Differentiate $f(x) = \dfrac{\cos(x^3 + 2)}{x + 1}$. (MA161 Paper 1, 2012/13)
- Suppose that $A(r)$ represents the area of a circle of radius r, i.e., $A(r) = \pi r^2$. The radius in turn can be expressed in terms of the circumference ℓ of the circle as $r(\ell) = \frac{\ell}{2\pi}$. How does the area change with respect to the circumference?

Chain Rule: Applications.

The Chain Rule allows us to deduce the Quotient Rule from the Product Rule.

First suppose that $g(x) = \dfrac{1}{v(x)}$.

Then $g(x) = u(v(x))$, where $u(v) = \frac{1}{v} = v^{-1}$ and $u'(v) = -v^{-2}$.

Chain Rule: $g'(x) = u'(v(x)) \cdot v'(x) = -v(x)^{-2} \cdot v'(x) = \dfrac{-v'(x)}{v(x)^2}$.

Now suppose that $f(x) = \dfrac{u(x)}{v(x)}$.

Reading this as a product, $f(x) = u(x) \cdot \dfrac{1}{v(x)}$.

And, by the Product Rule:

$$f'(x) = u'(x) \cdot \frac{1}{v(x)} + u(x) \cdot \frac{-v'(x)}{v(x)^2} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

Exercises.

1. Show that $\frac{d}{dx}x^4 = 4x^3$ from first principles.
2. Differentiate the following functions.
   (i) $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$.
   (ii) $f(x) = \dfrac{1+x}{1-x}$.
3. (MA161 Exam, Summer 2011/2012) Consider the piecewise defined function

$$f(x) = \begin{cases} x^2, & \text{if } x \leqslant 6, \\ qx + r, & \text{if } x > 6. \end{cases}$$

   For which values of q and r are both $f(x)$ and $f'(x)$ continuous at $x = 6$?
4. Use the limit laws to show that $(f(x) + g(x))' = f'(x) + g'(x)$.

   [Hint. Complete and justify $\lim_{h\to 0} \frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} = \cdots = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$.]

Exercises.

5. Differentiate $f(x) = \sin(x)\cos(x)$.
6. Differentiate $f(x) = e^{kx}$ for a constant $k \in \mathbb{R}$.
7. Differentiate the following functions.
   (a) $f(x) = 4x^3 + e^{4x}$.
   (b) $f(x) = x^2 \sin(x)$.
   (c) $f(x) = \cos(x)/x^3$.
   (d) $f(x) = \sqrt[3]{x^2 + 2x + 1}$.
   (e) $f(x) = (x^2 + 1)^6$.
8. It is possible to express $f(x) = e^x$ as the infinite series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n + \cdots,$$

where $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.
Use this expression to show that $\frac{d}{dx}e^x = e^x$.