Week 9: Differentiation Rules.

MA161/MA1161: Semester 1 Calculus.

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Assignment 2 is Online.

The **Midterm 2** assignment is **now online** at

`http://mathswork.nuigalway.ie/webwork2/1920-MA161`

with a deadline of Friday, November 22.

Remember:

- Students need to score at least 35% in the continuous assessment in order to complete the module.
- Assignments cannot and will not be repeated.

Need help?

- Go to SUMS.
- Go to tutorials.
- Ask questions at lectures.

Recall: The Derivative.

Geometrically, the **derivative** of a function f at a point p is the **slope of the tangent** to f at p.
The formal, mathematical definition is:

The derivative of a function $f\colon D \to \mathbb{R}$ at a point $p \in D$ is defined as

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h},$$

provided the limit exists.

The derivative of a function f at a point p is the instantaneous **rate of change** of f at p.
Derivatives can be computed **from first principles**, that is directly from the above definition.
Differentiation rules are formulas which simplify differentiation tasks, and apply to wide ranges of functions.

The Power Rule.

For $n \in \mathbb{N}$, we have $\frac{d}{dx}x^n = nx^{n-1}$.

$(x^2)' = 2x, \quad (x^3)' = 3x^2, \quad (x^4)' = 4x^3, \quad \ldots$

Proof. Let $f(x) = x^n$. By the **Binomial Theorem**,

$$f(x+h) = (x+h)^n = x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n,$$

whence $f(x+h) - f(x) = h\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}\right)$. So

$$\frac{f(x+h) - f(x)}{h} \to nx^{n-1} \text{ as } h \to 0.$$

□

This result extends to any real exponent a:

Power Rule

For $a \in \mathbb{R}$, we have $\frac{d}{dx}x^a = ax^{a-1}$.

$\left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2}; \quad (\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2}; \quad \ldots$

Differentiation Methods.

Suppose that $f(x)$ and $g(x)$ are differentiable functions and that c is a constant. Then the functions $c\,f$, $f+g$ and $f-g$ are differentiable, and we have

- $c' = 0$,
- $(cf)' = c\,f'$,
- $(f+g)' = f' + g'$,
- $(f-g)' = f' - g'$.

Example

Differentiate $f(x) = 3x^4 - 5x^2 + 7$.

Solution: $f'(x) = 3(x^4)' - 5(x^2)' + (7)' = 12x^3 - 10x$.

Example

Find the derivative of $f(x) = 5x^4 + 3\sqrt[3]{x} - \frac{2}{x}$.

Solution: $f'(x) = 5(x^4)' + 3(x^{1/3})' - 2(x^{-1})' = 20x^3 + x^{-2/3} + 2x^{-2}$.

Derivatives of Exponential Functions.

The derivative of an exponential function $f(x) = a^x$ is, remarkably, a constant multiple of the function $f(x)$ itself:

If $f(x) = a^x$ then $f'(x) = f'(0)\, a^x$.

Proof. As

$$f(x+h) - f(x) = a^{x+h} - a^x = a^x a^h - a^x = a^x(a^h - a^0),$$

we find the derivative of $f(x)$ as

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = a^x \lim_{h\to 0} \frac{a^h - a^0}{h} = a^x f'(0). \qquad \square$$

Euler's number

$$e = 2.718281828459045235360287471352662497757247093\ldots$$

has the property $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$. As a consequence:

$(e^x)' = e^x$.

Exponential Growth.

A population that grows at a rate that equals its current size can be modelled by an exponential function.

Example

Suppose that the rate of change of a certain population of bacteria is equal to the current population. Find a formula for the population at any given time.

Solution. Let $p(t)$ be the **size of the population** at time t.
The **growth rate** is the derivative $\frac{d}{dt}p(t) = p'(t)$.
The rate of growth is **equal** to the population size: $p'(t) = p(t)$.
If we set

$$p(t) = Ce^t$$

for some constant C, then $p'(t) = Ce^t = p(t)$.
Also, $C = Ce^0 = p(0)$ is the size of the population at time $t = 0$.

The Product Rule.

see Section 3.2 of the Book . . .

The derivative of a product uv of functions u and v is **not** the product of the derivatives of u and v, **but**:

Product Rule

If u and v are differentiable functions then

$$(uv)' = u'v + uv'.$$

Differentiate $f(x) = x^2 e^x$.

Solution. Read $f(x)$ as $u(x)\,v(x)$, where

$$u(x) = x^2 \textit{ and } v(x) = e^x.$$

Then $u'(x) = 2x$ and $v'(x) = e^x$. Hence

$$f'(x) = u(x)\,v'(x) + u'(x)\,v(x) = x^2 e^x + 2x\,e^x = (x^2 + 2x)e^x.$$

Applications of the Product Rule.

Use the product rule to differentiate $f(x) = x^3$.

$f(x) = x^3 = x \cdot x^2 = u(x)\,v(x)$, where $u(x) = x$ and $v(x) = x^2$.
Then $u'(x) = 1$ and $v'(x) = 2x$, and

$$f'(x) = u(x)\,v'(x) + u'(x)\,v(x) = x \cdot 2x + 1 \cdot x^2 = 3x^2.$$

Differentiate $f(x) = (2x^3 + 1)(3x - 2)$.

$u(x) = 2x^3 + 1 \implies u'(x) = 6x^2$ and $v(x) = 3x - 2 \implies v'(x) = 3$.

$$f'(x) = 3(2x^3 + 1) + 6x^2(3x - 2) = 24x^3 - 12x^2 + 3.$$

Example (MA161 Semester 1 Exam 2013/14)

Find the derivative of $f(x) = 4x^3 - x^2 e^x$.

$$\begin{aligned} f'(x) &= (4x^3)' - (x^2e^x)' = 12x^2 - ((x^2)(e^x)' + (x^2)'(e^x)) \\ &= 12x^2 - (x^2e^x + 2xe^x) = 12x^2 - (x^2 + 2x)e^x. \end{aligned}$$

The Quotient Rule.

Quotient Rule

If u and v are differentiable functions, and if $v \neq 0$ then

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

Differentiate $f(x) = e^x/x$.

Solution. Read $f(x)$ as $u(x)/v(x)$, where $u(x) = e^x$ and $v(x) = x$. Then $u'(x) = e^x$ and $v'(x) = 1$. Hence

$$f'(x) = (u'(x)\,v(x) - u(x)\,v'(x))/v(x)^2 = (e^x x - e^x)/x^2.$$

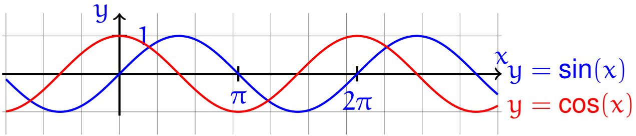
Differentiate $g(x) = (1 + x^2)/(1 + x)$.

$u(x) = 1 + x^2 \implies u'(x) = 2x$ and $v(x) = 1 + x \implies v'(x) = 1$.

$$f'(x) = (2x(1+x) - (1+x^2))/(1+x)^2 = (x^2+2x-1)/(1+x)^2.$$

Derivatives of Trigonometric Functions.

Recall: The graphs of $\sin(x)$ and $\cos(x)$ look like:



Given that the derivative of $f(x)$ at $x = p$ is the slope of the tangent to $f(x)$ at p, it is not hard to believe that:

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x).$$

Then, using the Quotient Rule, one finds that:

$$(\tan(x))' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

The Chain Rule.

In order to compute the derivative of the **composite function** $(u \circ v)(x) = u(v(x))$, one regards u **as a function of** v (and v as a function of x).

Then one determines the derivative $\frac{d}{dv}u$ of u (wrt. v) and the derivative $\frac{d}{dx}$ of v (wrt. x).

The derivative of $u(v(x))$ (wrt. x) then is the **product** of those two.

Chain Rule

If u and v are functions so that v is differentiable at x and u is differentiable at $v(x)$ then $u \circ v$ is differentiable at x and its derivative (wrt. x) is

$$(u \circ v)'(x) = u'(v(x)) \cdot v'(x).$$

The chain rule is sometimes written as

$$\frac{d}{dx}u = \frac{d}{dv}u \cdot \frac{d}{dx}v; \quad \text{or even as} \quad \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}.$$

The Chain Rule: Examples.

Differentiate $\sin(x^2)$.

Set $u(v) = \sin(v)$ and $v(x) = x^2$, then $\sin(x^2) = u(v(x))$.
We have $\frac{d}{dv}u(v) = (\sin(v))' = \cos(v)$ and $\frac{d}{dx}v(x) = (x^2)' = 2x$.
Hence $(\sin(x^2))' = u'(v(x)) \cdot v'(x) = \cos(x^2) \cdot 2x$.

Examples

- Find $f'(x)$ when $f(x) = \sqrt{x^2 + 1}$. (Source: Stewart, p. 198)
- Differentiate $f(x) = (\sin(2x))^6$. (Apply the Chain Rule repeatedly.)
- Differentiate $f(x) = \dfrac{\cos(x^3 + 2)}{x + 1}$. (MA161 Paper 1, 2012/13)
- Suppose that $A(r)$ represents the area of a circle of radius r, i.e., $A(r) = \pi r^2$. The radius in turn can be expressed in terms of the circumference ℓ of the circle as $r(\ell) = \frac{\ell}{2\pi}$. How does the area change with respect to the circumference?

Chain Rule: Applications.

The Chain Rule allows us to deduce the Quotient Rule from the Product Rule.

First suppose that $g(x) = \dfrac{1}{v(x)}$.

Then $g(x) = u(v(x))$, where $u(v) = \frac{1}{v} = v^{-1}$ and $u'(v) = -v^{-2}$.

Chain Rule: $g'(x) = u'(v(x)) \cdot v'(x) = -v(x)^{-2} \cdot v'(x) = \dfrac{-v'(x)}{v(x)^2}$.

Now suppose that $f(x) = \dfrac{u(x)}{v(x)}$.

Reading this as a product, $f(x) = u(x) \cdot \dfrac{1}{v(x)}$.

And, by the Product Rule:

$$f'(x) = u'(x) \cdot \frac{1}{v(x)} + u(x) \cdot \frac{-v'(x)}{v(x)^2} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

Exercises.

1. Show that $\frac{d}{dx}x^4 = 4x^3$ from first principles.
2. Differentiate the following functions.
   (i) $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$.
   (ii) $f(x) = \dfrac{1+x}{1-x}$.
3. (MA161 Exam, Summer 2011/2012) Consider the piecewise defined function

$$f(x) = \begin{cases} x^2, & \text{if } x \leqslant 6, \\ qx + r, & \text{if } x > 6. \end{cases}$$

   For which values of q and r are both $f(x)$ and $f'(x)$ continuous at $x = 6$?
4. Use the limit laws to show that $(f(x) + g(x))' = f'(x) + g'(x)$.

   [Hint. Complete and justify $\lim_{h\to 0} \frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} = \cdots = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$.]

Exercises.

5. Differentiate $f(x) = \sin(x)\cos(x)$.
6. Differentiate $f(x) = e^{kx}$ for a constant $k \in \mathbb{R}$.
7. Differentiate the following functions.
   (a) $f(x) = 4x^3 + e^{4x}$.
   (b) $f(x) = x^2 \sin(x)$.
   (c) $f(x) = \cos(x)/x^3$.
   (d) $f(x) = \sqrt[3]{x^2 + 2x + 1}$.
   (e) $f(x) = (x^2 + 1)^6$.
8. It is possible to express $f(x) = e^x$ as the infinite series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n + \cdots,$$

   where $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.
   Use this expression to show that $\frac{d}{dx}e^x = e^x$.