Week 8: From Tangents to Derivatives.

MA161/MA1161: Semester 1 Calculus.

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Tangents.

see Section 2.7 of the Book . . .

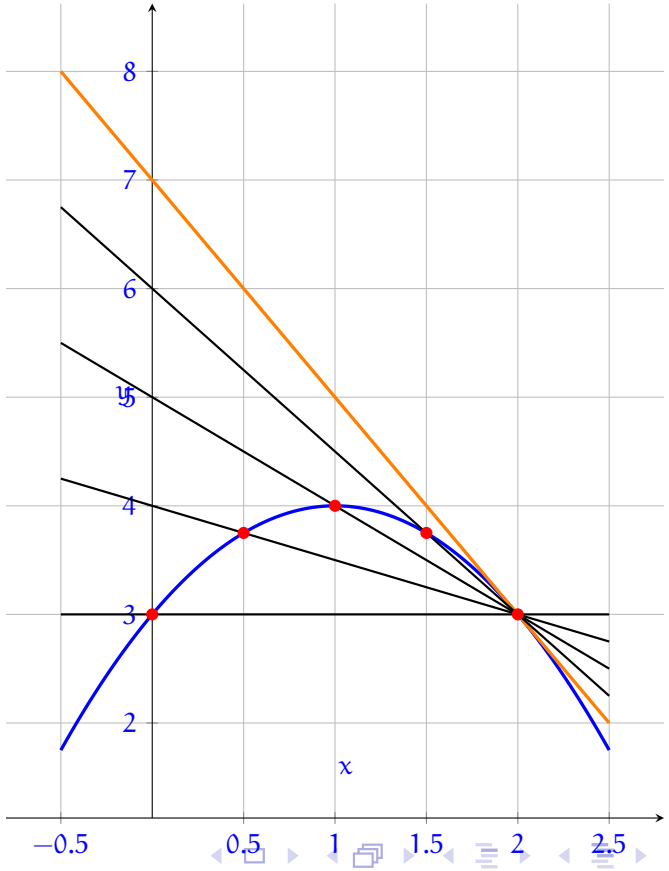
The **tangent** of a function $f\colon D \to \mathbb{R}$ at a point p is the straight line passing through the point $(p, f(p))$ in the **same direction** as f. What is this direction? And how do we find it?

Recall: The **slope** of a line that intersects the graph of f at two distinct points x and p is

$$\frac{f(x) - f(p)}{x - p}.$$

As $x \to p$, this line approaches the tangent of f at p, and its slope approaches the slope of the tangent, which therefore is the **limit**

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}.$$



The Derivative.

Geometrically, the **derivative** of a function f at a point p is the **slope of the tangent** to f at p.
More formally (using $x = p + h$ and $x \to p$ as $h \to 0$):

Definition

The derivative of a function $f\colon D \to \mathbb{R}$ at a point p is defined as

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h},$$

provided the limit exists.

Example (Case $f(x) = x^2$ and $p = 1$.)

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} (2 + h) = 2. \end{aligned}$$

The Derivative as a Function.

see Section 2.8 of the Book . . .

The derivative $f'(p)$ of a function f is defined at all points p, where the the limit of the quotient of differences is defined.

So we can regard f' as a function, the **derivative of** f, which is defined by the rule

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Notation

There are several ways to write the derivative of f. We'll mostly use f', and sometimes $\frac{d}{dx} f$, to emphasize the variable x.

Differentiation From First Principles.

Calculating the (function that is the) derivative of a given function f by using the definition of the derivative is called **differentiation from first principles**.

Example (MA161 2012/13 Semester 1 Exam)

Use that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to show the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x. \end{aligned}$$

Examples.

Find the derivative of $f(x) = \frac{1}{x}$ from first principles

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)h}$$
$$= \lim_{h\to 0} \frac{-h}{x(x+h)h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

Find the derivative of $f(x) = \sqrt{x}$ from first principles

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$
$$= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

Hence $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$

The Meaning of Derivatives.

Differential Calculus has its origin in Physics.
The slope of the tangent to f at p, i.e., the derivative of f at p, is the instantaneous **rate of change** of f at p.

Examples

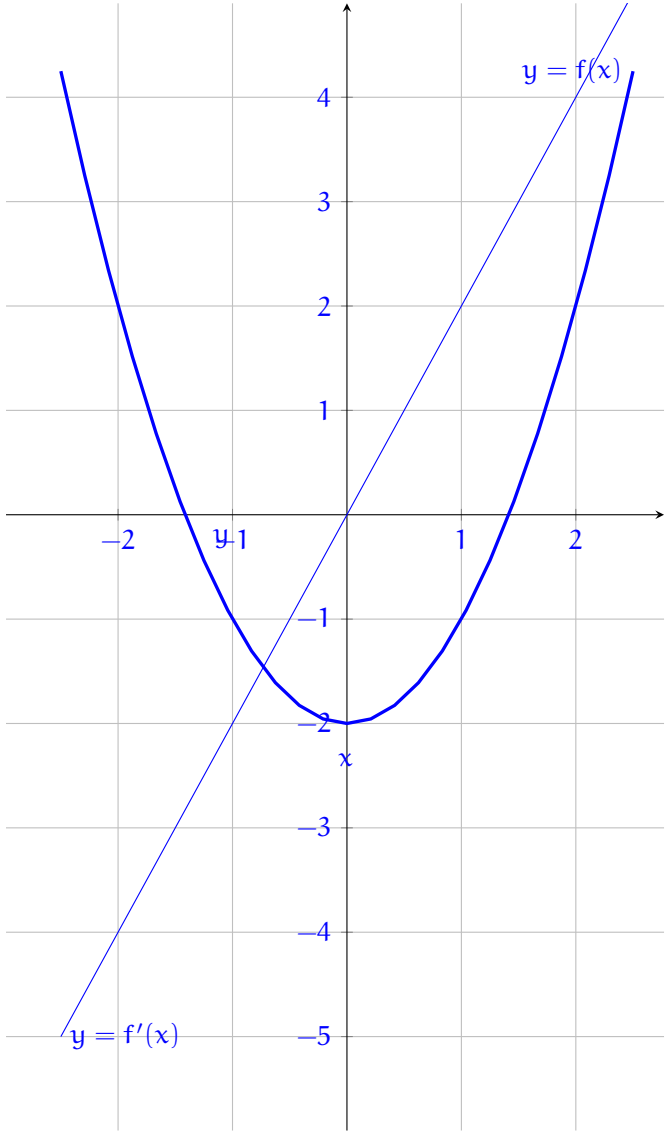
$f(x)$	$f'(x)$
distance from a point	velocity
velocity at a point	acceleration
work at time x	power
concentration of a chemical	rate of reaction
cost of producing x	marginal cost
...	...

The Derivative and the Graph of a Function.

- The function $f(x) = x^2 - 2$ has the function $f'(x) = 2x$ as its derivative.

The derivative f' of a function f reveals **properties of the graph** $y = f(x)$:

- If $f'(x) < 0$ then $f(x)$ is **decreasing**.
- If $f'(x) > 0$ then $f(x)$ is **increasing**.
- And if $f'(x) = 0$ then $f(x)$ possibly is a **local minimum** or **maximum** ...



Exercises.

1. Let $f(x) = 7 - 13x$. Use the definition of the derivative to find $f'(-2)$. What is the derivative f' as a function of x?
2. Let $f(x) = x^3$. Use the definition of that derivative to show that $f'(x) = 3x^2$.
3. Use the trigonometric identity

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

   to show that $\frac{d}{dx}\sin(x) = \cos(x)$.

   [Hint. $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.]
4. Use the trigonometric identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

   to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.
5. What is the equation of the tangent to $f(x) = x^2 + 4x + 3$ at $x = 1$?