Week 8: From Tangents to Derivatives.

MA161/MA1161: Semester 1 Calculus.

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Assignments.

Problem Set 3 is due for submission on Friday, November 4, before 5pm.

Remember: MA161 students need to score at least 35% in these assignments in order to complete the module.

That's an average of 5.25 points (out of 15) on each of the (four) problem sets.

Assignments cannot and will not be repeated.
Need help?

- Go to SUMS.
- Go to tutorials.
- Ask questions at lectures.

Tangents.

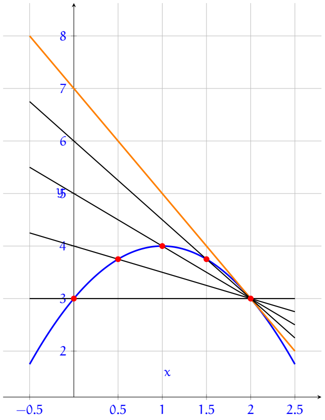
[Section 2.7 of the Book]
The **tangent** of a function $f \colon D \to \mathbb{R}$ at a point p is the straight line passing through the point $(p, f(p))$ in the **same direction** as f.
What is the direction of a function f at a point p?
It's the **slope of the tangent**, which is a limit as follows.

The slope of a line that intersects the graph of f at two distinct points x and p is

$$\frac{f(x) - f(p)}{x - p}.$$

As $x \to p$, this line approaches the tangent of f at p, and its slope approaches the slope of the tangent, which therefore is the **limit**

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}.$$



The Derivative.

Geometrically, the **derivative** of a function f at a point p is the **slope of the tangent** to f at p.
More formally (using $x = p + h$ and $x \to p$ as $h \to 0$):

Definition

The derivative of a function $f\colon D \to \mathbb{R}$ at a point p is defined as

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h},$$

provided the limit exists.

Example (Assume $f(x) = x^2$ and $p = 1$.)

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} (2 + h) = 2. \end{aligned}$$

The Derivative as a Function.

[Section 2.8 of the Book]
The derivative $f'(p)$ of a function f is defined at all points p, where the the limit of the quotient of differences is defined.

So we can regard f' as a function, the **derivative of f**, which is defined by the rule

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Notation

There are several ways to write the derivative of f. We'll mostly use f', and sometimes $\frac{d}{dx}f$, to emphasize the variable x.

Differentiation From First Principles.

Calculating the (function that is the) derivative of a given function f by using the definition of the derivative is called **differentiation from first principles**.

Example (MA161 2012/13 Semester 1 Exam)

Use that $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to show the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

Examples.

Find the derivative of $f(x) = \frac{1}{x}$ from first principles

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)h}$$

$$= \lim_{h\to 0} \frac{-h}{x(x+h)h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

Find the derivative of $f(x) = \sqrt{x}$ from first principles

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

Hence $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$.

The Meaning of Derivatives.

Differential Calculus has its origin in Physics.
The slope of the tangent to f at p, i.e., the derivative of f at p, is the instantaneous **rate of change** of f at p.

Examples

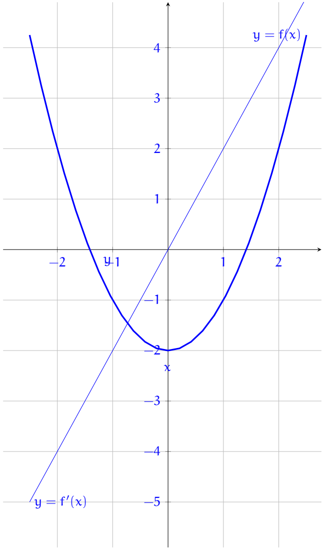
$f(x)$	$f'(x)$
distance from a point	velocity
velocity at a point	acceleration
work at time x	power
concentration of a chemical	rate of reaction
cost of producing x	marginal cost
...	...

The Derivative and the Graph of a Function.

- Recall: The function $f(x) = x^2 - 2$ has the function $f'(x) = 2x$ as its derivative.

The derivative of a function f reveals properties of the function graph $y = f(x)$:

- If $f'(x) < 0$ then $f(x)$ is **decreasing**.
- If $f'(x) > 0$ then $f(x)$ is **increasing**.



The Power Rule.

For $n \in \mathbb{N}$, we have $\frac{d}{dx}x^n = nx^{n-1}$.

$(x^2)' = 2x, \quad (x^3)' = 3x^2, \quad (x^4)' = 4x^3, \quad \ldots$

Proof. Let $f(x) = x^n$. By the **Binomial Theorem**,

$$f(x+h) = (x+h)^n = x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n,$$

whence $f(x+h) - f(x) = h\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}\right)$. So

$$\frac{f(x+h) - f(x)}{h} \to nx^{n-1} \text{ as } h \to 0.$$

$\square$

This result extends to any real exponent a:

Power Rule

For $a \in \mathbb{R}$, we have $\frac{d}{dx}x^a = ax^{a-1}$.

$\left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2}; \quad (\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2}; \quad \ldots$

Differentiation Methods.

Suppose that $f(x)$ and $g(x)$ are differentiable functions and that c is a constant. Then the functions $c\,f$, $f+g$ and $f-g$ are differentiable, and we have

- $c' = 0$,
- $(cf)' = c\,f'$,
- $(f+g)' = f' + g'$,
- $(f-g)' = f' - g'$.

Example

Differentiate $f(x) = 3x^4 - 5x^2 + 7$.

Solution: $f'(x) = 3(x^4)' - 5(x^2)' + (7)' = 12x^3 - 10x$.

Example

Find the derivative of $f(x) = 5x^4 + 3\sqrt[3]{x} - \frac{2}{x}$.

Solution: $f'(x) = 5(x^4)' + 3(x^{1/3})' - 2(x^{-1})' = 20x^3 + x^{-2/3} + 2x^{-2}$.

Derivatives of Exponential Functions.

The derivative of an exponential function $f(x) = a^x$ is, remarkably, a constant multiple of the function $f(x)$ itself:

If $f(x) = a^x$ then $f'(x) = f'(0)\, a^x$.

Proof. As

$$f(x+h) - f(x) = a^{x+h} - a^x = a^x a^h - a^x = a^x(a^h - a^0),$$

we find the derivative of $f(x)$ as

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = a^x \lim_{h\to 0} \frac{a^h - a^0}{h} = a^x\, f'(0). \qquad \square$$

Euler's number

$$e = 2.718281828459045235360287471352662497757247093\ldots$$

has the property $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = 1$. As a consequence:

$(e^x)' = e^x$.

Exponential Growth.

A population that grows at a rate that equals its current size can be modelled by an exponential function.

Example

Suppose that the rate of change of a certain population of bacteria is equal to the current population. Find a formula for the population at any given time.

Solution. Let $p(t)$ be the **size of the population** at time t.
The **growth rate** is the derivative $\frac{d}{dt}p(t) = p'(t)$.
The rate of growth is **equal** to the population size: $p'(t) = p(t)$.
If we set

$$p(t) = Ce^t$$

for some constant C, then $p'(t) = Ce^t = p(t)$.
Also, $C = Ce^0 = p(0)$ is the size of the population at time $t = 0$.

Exercises.

1. Let $f(x) = 7 - 13x$. Use the definition of the derivative to find $f'(-2)$. What is the derivative f' as a function of x?
2. Let $f(x) = x^3$. Use the definition of that derivative to show that $f'(x) = 3x^2$.
3. Use the trigonometric identity

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

   to show that $\frac{d}{dx}\sin(x) = \cos(x)$.

   [Hint. $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.]
4. Use the trigonometric identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

   to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.
5. What is the equation of the tangent to $f(x) = x^2 + 4x + 3$ at $x = 1$?