Week 7: Limits at Infinity.

MA161/MA1161: Semester 1 Calculus.

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Continuous Assessment.

Midterm 1 is due for submission this Tuesday.

- CA accounts for 40% of the overall mark.
- Midterm is worth 10% of those 40%.
- You need a score of at least 14% of those 40% to complete the module.
- Midterms cannot and will not be repeated.

Need help?

- Go to tutorials.
- Go to SUMS.
- Ask questions at or after lectures.

Recall: Limits.

The idea of a limit

We say that the **limit of a function f at the point p is L** and write

$$\lim_{x \to p} f(x) = L,$$

if we can make $f(x)$ as close to L as we would like, by taking x as close to p as is necessary.

For the limit $\lim_{x \to p} f(x)$ to exist it is **not necessary** that the function f is defined at $x = p$. All that matters is how $f(x)$ is defined for x **near** p.

Example

Consider the function $f(x) = \dfrac{2x^2 - 2x}{x - 1}$. Here $f(x)$ is not defined at $x = 1$ (as this would involve an illegal division by zero). But if $x \neq 1$ then $\dfrac{2x^2 - 2x}{x - 1} = \dfrac{2x(x-1)}{x-1} = 2x$ and thus $\lim_{x \to 1} f(x) = \lim_{x \to 1} 2x = 2$.

Recall: Continuity.

see Section 2.5 of the Book . . .

We have seen that the limit of a function $f(x)$ as x approaches p can often be found by simply calculating the value of f at p. In such a situation we say that "f is **continuous** at p".

A function $f\colon D \to \mathbb{R}$ is **continuous** at a point $p \in D$ if

$$\lim_{x \to p} f(x) = f(p).$$

Note that this definition of continuity implicitly requires that

- $f(p)$ is defined (that is, p is in the domain of f),
- the limit $\lim_{x \to p} f(x)$ exists,
- and $\lim_{x \to p} f(x) = f(p)$.

Examples of Lack of Continuity.

- The function
$$f(x) = \frac{x^2 - x - 2}{x - 2}$$
is **not continuous** at $x = 2$, as $f(2)$ is not defined.
- The function
$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$
is **not continuous** at $x = 0$, as $\lim_{x \to 0} f(x)$ does not exist.
- The function
$$f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2}, & x \neq 2, \\ 1, & x = 2, \end{cases}$$
is **not continuous** at $x = 2$, as $\lim_{x \to 2} f(x) = 3 \neq 1 = f(x)$.

Continuity on an Interval.

We say that a function $f\colon D \to \mathbb{R}$ is **continuous**, if it is **continuous at all points in its domain** D.
If the domain D consists of intervals, this means that the function needs to be continuous on each interval, in the following sense.

A function $f\colon D \to \mathbb{R}$ is **continuous on an interval** $I \subseteq D$ if it is continuous at every point p in the interval I.

Here, one-sided limits are used for the end points of intervals.

One-Sided Continuity.

A function $f\colon D \to \mathbb{R}$ is continuous from the left (or from the right) at a point $p \in D$ if

$$\lim_{x\to p^-} f(x) = f(p) \quad \text{(or } \lim_{x\to p^+} f(x) = f(p)\text{).}$$

Geometrically, you can think of a continuous function as one whose graph can be drawn without lifting the pen from the paper.

Examples.

Example

For which values of a, if any, is the following function f continuous on all of $\mathbb{R}$?

$$f(x) = \begin{cases} 4 - ax, & \text{if } x < 1, \\ ax^2 + x + 1, & \text{if } x \geqslant 1. \end{cases}$$

Example (MA161 Paper 1 2012/13)

Find the two values of b for which the following function f is continuous at $x = 2$.

$$f(x) = \begin{cases} -2x + b, & x < 2, \\ -24/(x - b), & x \geqslant 2. \end{cases}$$

Most of our Functions are Continuous.

This is a consequence of the **Limit Laws**: If the functions f and g are continuous at the point $p \in \mathbb{R}$, and if $c \in \mathbb{R}$ is any constant, then the following functions are **also continuous** at a:

1. $f + g$; 2. $f - g$; 3. $c\,f$; 4. fg; 5. f/g if $g(a) \neq 0$.

Moreover, if g is continuous at a and if f is continuous at $g(a)$ then the composite function $f \circ g$ is continuous at a.

Hence, most of our functions are continuous most of the time.

The following types of functions are **continuous at every point in their domain**:

- polynomials and rational functions;
- power and root functions;
- trigonometric functions and their inverses;
- exponential and logarithmic functions.

Red Squirrels.

A Motivating Example.

A team of researchers studying red squirrels in Ireland has determined that the population density (i.e., the number of squirrels per square kilometre) in t years from now can be modelled as

$$P(t) = \frac{200t^2 + 50t + 900}{t^3 + 30}.$$

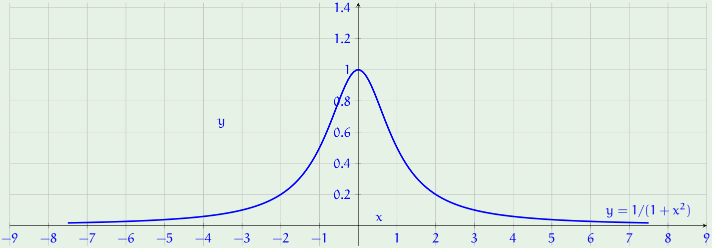
1. What is the current population density?
2. What will the population density be in 5 years time?
3. What can be predicted about the **long-term** population density?

The first and the second question are relatively easy to answer: simply substitute $t = 0$, or $t = 5$ in $P(t)$.
In order to answer the third question, we need to apply the concept of a **limit at infinity**.

For Example.

So far, we have studied the limit of $f(x)$ as x approaches some point p on the x-axis.
The idea of a limit can be extended to the study of the behavior of $f(x)$ as x gets very large.

Consider the graph of $f(x) = \dfrac{1}{1+x^2}$.



We see that $f(x)$ tends to 0 as x goes to ∞, or to $-\infty$.

Limits at Infinity.

Limit at $+\infty$ and $-\infty$.

We write

$$\lim_{x\to\infty} f(x) = L \qquad \text{(or } \lim_{x\to-\infty} f(x) = L\text{)},$$

and say that the limit of $f(x)$ as x approaches $+\infty$ (or $-\infty$) is L, if we can make $f(x)$ as close to L as we would like, by simply choosing x as a sufficiently large positive (or negative) number.

This may also be written as

$$f(x) \to L \text{ as } x \to \infty \text{ (or } x \to -\infty\text{)}.$$

In either case, the line $y = L$ is called a **horizontal asymptote** of $f(x)$.

Basic Examples.

Many limits at infinity can be reduced to these three:

- $f(x) = 1$. The value of $f(x)$ will always be 1, no matter how large we choose x:

$$\lim_{x\to\infty} 1 = 1 \quad \text{and} \quad \lim_{x\to-\infty} 1 = 1.$$

- $f(x) = \frac{1}{x}$. The larger we choose x, the closer will $f(x)$ be to 0:

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

- $f(x) = x$. The larger we choose x, the larger will $f(x)$ be:

$$\lim_{x\to\infty} x = \infty \quad \text{and} \quad \lim_{x\to-\infty} x = -\infty.$$

(The limit at infinity can be infinite.)

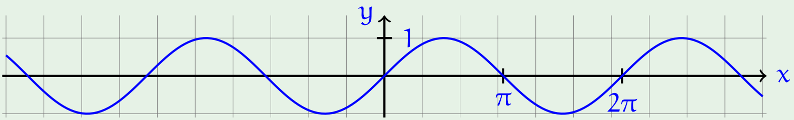
DNE.

The limit at infinity need not exist.

Example ($f(x) = \sin(x)$.)

$$\lim_{x\to\infty} \sin(x) \text{ and } \lim_{x\to-\infty} \sin(x)$$

do not exist, as not matter how large we choose x, the value of $\sin(x)$ keeps oscillating between -1 and 1.



Neither does $\lim_{x\to\infty} \cos(x)$ exist, nor $\lim_{x\to\infty} \tan(x)$ . . .

Limit Laws at Infinity.

The usual Limit Laws apply to limits at infinity;
the limit of a sum is the sum of the limits . . . :

Limit Laws for Limits at ∞ (or $-\infty$).

Suppose that $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are functions and that the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist. Then:

1. $\lim_{x\to\infty} \big(f(x)+g(x)\big) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$.
2. $\lim_{x\to\infty} \big(f(x)-g(x)\big) = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$.
3. $\lim_{x\to\infty} \big(c\, f(x)\big) = c \lim_{x\to\infty} f(x)$ for all constants $c \in \mathbb{R}$.
4. $\lim_{x\to\infty} \big(f(x)g(x)\big) = \Big(\lim_{x\to\infty} f(x)\Big)\Big(\lim_{x\to\infty} g(x)\Big)$.
5. $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$, provided that $\lim_{x\to\infty} g(x) \neq 0$.

Examples.

Using the limit laws, we can compute the following limits.

$$\begin{aligned}
\lim_{x\to\infty} \frac{1-x}{x} &= \lim_{x\to\infty}\left(\frac{1}{x}-1\right) \overset{(2.)}{=} \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty} 1 \\
&= 0 - 1 = -1. \\
\lim_{x\to\infty} \frac{1}{x^2} &= \lim_{x\to\infty}\left(\frac{1}{x}\cdot\frac{1}{x}\right) \overset{(4.)}{=} \left(\lim_{x\to\infty}\frac{1}{x}\right)\left(\lim_{x\to\infty}\frac{1}{x}\right) \\
&= 0 \cdot 0 = 0. \\
\lim_{x\to\infty} \frac{1-x}{x^2} &= \lim_{x\to\infty}\left(\frac{1}{x^2}-\frac{1}{x}\right) \overset{(2.)}{=} \lim_{x\to\infty}\frac{1}{x^2} - \lim_{x\to\infty}\frac{1}{x} \\
&= 0 - 0 = 0. \\
\lim_{x\to\infty} \frac{1}{x^n} &= \lim_{x\to\infty}\left(\frac{1}{x^{n-1}}\cdot\frac{1}{x}\right) \overset{(4.)}{=} \left(\lim_{x\to\infty}\frac{1}{x^{n-1}}\right)\left(\lim_{x\to\infty}\frac{1}{x}\right) \\
&= 0 \cdot 0 = 0, \text{ for all } n > 1.
\end{aligned}$$

Limits of Rational Functions.

To compute the limit of a rational function at infinity:

- divide **both numerator and denominator** by the highest power of x **in the denominator**;
- then apply the limit laws ...

Example (From the Book, Section 2.6.)

Evaluate the limit of $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ as $x \to \infty$.

Soln. $\dfrac{(3x^2 - x - 2)/x^2}{(5x^2 + 4x + 1)/x^2} = \dfrac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + 4\frac{1}{x} + \frac{1}{x^2}} \to \dfrac{3 - 0 - 0}{5 + 0 + 0} = \dfrac{3}{5}$, as $x \to \infty$.

Example (MA161 Paper 1, 2012/2013.)

Evaluate $\displaystyle\lim_{x\to\infty} \frac{x^4 - 4x^2 + 4}{x^5 + 5x^3}$.

Solution:

$$\frac{(x^4 - 4x^2 + 4)/x^5}{(x^5 + 5x^3)/x^5} = \frac{\frac{1}{x} - 4\frac{1}{x^3} + 4\frac{1}{x^5}}{1 + 5\frac{1}{x^2}} \to \frac{0 + 0 + 0}{1 + 0} = \frac{0}{1} = 0, \text{ as } x \to \infty.$$

Squirrels Revisted.

Example.

A team of researchers studying red squirrels in Ireland has determined that the population density (i.e., the number of squirrels per square kilometre) in t years from now can be modelled as

$$P(t) = \frac{200t^2 + 50t + 900}{t^3 + 30}.$$

1. What is the current population density? $P(0) = \frac{900}{30} = 30$.
2. What will the population density be in 3 years time? $P(3) = \frac{200 \cdot 3^2 + 50 \cdot 3 + 900}{3^3 + 30} = 50$.
3. What can be predicted about the **long-term** population density? $\frac{(200t^2 + 50t + 900)/t^3}{(t^3 + 30)/t^3} \to 0$ as $t \to \infty$: over time, the red squirrels will disappear. ☹

Exercises.

1. For what values of k is the function $f(x)$ continuous at $x = 2$?

$$f(x) = \begin{cases} 3x^4 - 5x^3 + x + 3, & \text{if } x < 2, \\ kx^2 + 3x - 4, & \text{if } x \geqslant 2. \end{cases}$$

2. Evaluate the following limits.
   (i) $\lim_{x \to \infty} 2^x$.
   (ii) $\lim_{x \to -\infty} 2^x$.
3. Evaluate the following limits.
   (i) $\lim_{x \to \infty} \dfrac{2x - 9x^3}{2x + 3x^3}$.
   (ii) $\lim_{x \to \infty} \dfrac{x^5 - x^4 + x^3 + x^2}{x^4 + 23\,112}$.
   (iii) $\lim_{x \to \infty} \dfrac{12x - 18x^2 + 6}{8x + x^3 + 16}$.
4. New research on the red squirrel has called into question the population model used above. The team **now** believes the model should be

$$P(t) = \frac{200t^2 + 50t + 900}{t + 30}.$$

What is the predicted long-term population density?