Week 7: Limits at Infinity.

MA161/MA1161: Semester 1 Calculus.

Prof. Götz Pfeiffer

School of Mathematics, Statistics and Applied Mathematics
NUI Galway

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Assignments.

Problem Set 2 is due for submission this Friday before 5pm.

Remember: MA161 students need to score at least 35% in these assignments in order to complete the module.

That's an average of 5.25 points (out of 15) on each of the (four) problem sets.

Assignments cannot and will not be repeated.
Need help?

- Go to SUMS.
- Go to tutorials.
- Ask questions at or after lectures.

Recall: Limits.

The idea of a limit

We say that the **limit of a function f at the point p is L** and write

$$\lim_{x \to p} f(x) = L,$$

if we can make $f(x)$ as close to L as we would like, by taking x as close to p as is necessary.

For the limit $\lim_{x \to p} f(x)$ to exist it is **not necessary** that the function f is defined at $x = p$. All that matters is how $f(x)$ is defined for x **near** p.

Example

Consider the function $f(x) = \dfrac{2x^2 - 2x}{x - 1}$. Here $f(x)$ is not defined at $x = 1$ (as this would involve an illegal division by zero). But if $x \neq 1$ then $\dfrac{2x^2 - 2x}{x - 1} = \dfrac{2x(x-1)}{x-1} = 2x$ and thus $\lim_{x \to 1} f(x) = \lim_{x \to 1} 2x = 2$.

Recall: Continuity.

[Section 2.5 of the Book]
We have seen that the limit of a function $f(x)$ as x approaches p can often be found by simply calculating the value of f at p. In such a situation we say that "f is **continuous** at p".

A function $f\colon D \to \mathbb{R}$ is **continuous** at a point $p \in D$ if

$$\lim_{x \to p} f(x) = f(p).$$

Note that this definition of continuity implicitly requires that

- $f(p)$ is defined (that is, p is in the domain of f),
- the limit $\lim_{x \to p} f(x)$ exists,
- and $\lim_{x \to p} f(x) = f(p)$.

Examples of Lack of Continuity.

- The function
$$f(x) = \frac{x^2 - x - 2}{x - 2}$$
is **not continuous** at $x = 2$, as $f(2)$ is not defined.
- The function
$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$
is **not continuous** at $x = 0$, as $\lim_{x \to 0} f(x)$ does not exist.
- The function
$$f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2}, & x \neq 2, \\ 1, & x = 2, \end{cases}$$
is **not continuous** at $x = 2$, as $\lim_{x \to 2} f(x) = 3 \neq 1 = f(x)$.

Continuity on an Interval.

We say that a function $f\colon D \to \mathbb{R}$ is **continuous**, if it is **continuous at all points in its domain** D.
If the domain D consists of intervals, this means that the function needs to be continuous on each interval, in the following sense.

A function $f\colon D \to \mathbb{R}$ is **continuous on an interval** $I \subseteq D$ if it is continuous at every point p in the interval I.

Here, one-sided limits are used for the end points of intervals.

One-Sided Continuity.

A function $f\colon D \to \mathbb{R}$ is continuous from the left (or from the right) at a point $p \in D$ if

$$\lim_{x \to p^-} f(x) = f(p) \quad (\text{or } \lim_{x \to p^+} f(x) = f(p)).$$

Geometrically, you can think of a continuous function as one whose graph can be drawn without lifting the pen from the paper.

Examples.

Example (Similar to Q15 on Problem Set 2)

For which values of a is the following function f continuous on all of $\mathbb{R}$?

$$f(x) = \begin{cases} 4 - ax, & \text{if } x < 1, \\ ax^2 + x + 1, & \text{if } x \geqslant 1. \end{cases}$$

Example (MA161 Paper 1 2012/13)

Find the two values of b for which the following function f is continuous at $x = 2$.

$$f(x) = \begin{cases} -2x + b, & x < 2, \\ -24/(x - b), & x \geqslant 2. \end{cases}$$

Most of our Functions are Continuous.

This is a consequence of the **Limit Laws**: If the functions f and g are continuous at the point $p \in \mathbb{R}$, and if $c \in \mathbb{R}$ is any constant, then the following functions are **also continuous** at a:

1. $f + g$; 2. $f - g$; 3. $c\,f$; 4. fg; 5. f/g if $g(a) \neq 0$.

Moreover, if g is continuous at a and if f is continuous at $g(a)$ then the composite function $f \circ g$ is continuous at a.

Hence, most of our functions are continuous most of the time.

The following types of functions are **continuous at every point in their domain**:

- polynomials and rational functions;
- power and root functions;
- trigonometric functions and their inverses;
- exponential and logarithmic functions.

Red Squirrels.

A Motivating Example.

A team of researchers studying red squirrels in Ireland has determined that the population density (i.e., the number of squirrels per square kilometre) in t years from now can be modelled as

$$P(t) = \frac{200t^2 + 50t + 900}{t^3 + 30}.$$

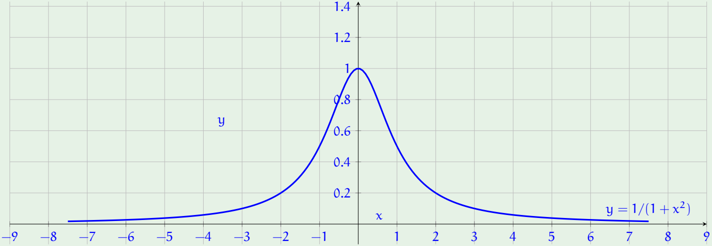
1. What is the current population density?
2. What will the population density be in 5 years time?
3. What can be predicted about the **long-term** population density?

The first and the second question are relatively easy to answer: simply substitute $t = 0$, or $t = 5$ in $P(t)$.
In order to answer the third question, we need to apply the concept of a **limit at infinity**.

For Example.

So far, we have studied the limit of $f(x)$ as x approaches some point p on the x-axis.
The idea of a limit can be extended to the study of the behavior of $f(x)$ as x gets very large.

Consider the graph of $f(x) = \dfrac{1}{1+x^2}$.



We see that $f(x)$ tends to 0 as x goes to ∞, or to $-\infty$.

Limits at Infinity.

Limit at $+\infty$ and $-\infty$.

We write

$$\lim_{x\to\infty} f(x) = L \qquad \text{(or } \lim_{x\to-\infty} f(x) = L\text{)},$$

and say that the limit of $f(x)$ as x approaches $+\infty$ (or $-\infty$) is L, if we can make $f(x)$ as close to L as we would like, by simply choosing x as a sufficiently large positive (or negative) number.

This may also be written as

$$f(x) \to L \text{ as } x \to \infty \text{ (or } x \to -\infty\text{)}.$$

In either case, the line $y = L$ is called a **horizontal asymptote** of $f(x)$.

Basic Examples.

Many limits at infinity can be reduced to these three:

- $f(x) = 1$. The value of $f(x)$ will always be 1, no matter how large we choose x:

$$\lim_{x \to \infty} 1 = 1 \quad \text{and} \quad \lim_{x \to -\infty} 1 = 1.$$

- $f(x) = \frac{1}{x}$. The larger we choose x, the closer will $f(x)$ be to 0:

$$\lim_{x \to \infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x} = 0.$$

- $f(x) = x$. The larger we choose x, the larger will $f(x)$ be:

$$\lim_{x \to \infty} x = \infty \quad \text{and} \quad \lim_{x \to -\infty} x = -\infty.$$

(The limit at infinity can be infinite.)

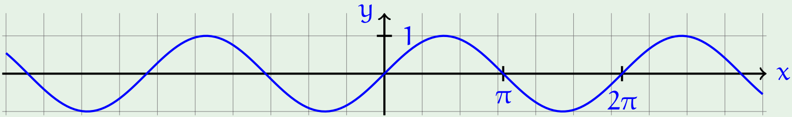
DNE.

The limit at infinity need not exist.

Example ($f(x) = \sin(x)$.)

$$\lim_{x\to\infty} \sin(x) \text{ and } \lim_{x\to-\infty} \sin(x)$$

do not exist, as not matter how large we choose x, the value of $\sin(x)$ keeps oscillating between -1 and 1.



Neither does $\lim_{x\to\infty} \cos(x)$ exist, nor $\lim_{x\to\infty} \tan(x)$ ...

Limit Laws at Infinity.

The usual Limit Laws apply to limits at infinity;
the limit of a sum is the sum of the limits ...:

Limit Laws for Limits at ∞ (or $-\infty$).

Suppose that $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are functions and that the limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$ exist. Then:

1. $\lim\limits_{x\to\infty} \big(f(x) + g(x)\big) = \lim\limits_{x\to\infty} f(x) + \lim\limits_{x\to\infty} g(x)$.
2. $\lim\limits_{x\to\infty} \big(f(x) - g(x)\big) = \lim\limits_{x\to\infty} f(x) - \lim\limits_{x\to\infty} g(x)$.
3. $\lim\limits_{x\to\infty} \big(c\, f(x)\big) = c \lim\limits_{x\to\infty} f(x)$ for all constants $c \in \mathbb{R}$.
4. $\lim\limits_{x\to\infty} \big(f(x)g(x)\big) = \Big(\lim\limits_{x\to\infty} f(x)\Big)\Big(\lim\limits_{x\to\infty} g(x)\Big)$.
5. $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to\infty} f(x)}{\lim\limits_{x\to\infty} g(x)}$, provided that $\lim\limits_{x\to\infty} g(x) \neq 0$.

Examples.

Using the limit laws, we can compute the following limits.

$$\lim_{x\to\infty} \frac{1-x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{x}{x}}{\frac{x}{x}} = \lim_{x\to\infty}\left(\frac{1}{x} - 1\right) \stackrel{(2.)}{=} \lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} 1$$
$$= 0 - 1 = -1.$$

$$\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty}\left(\frac{1}{x} \cdot \frac{1}{x}\right) \stackrel{(4.)}{=} \left(\lim_{x\to\infty} \frac{1}{x}\right)\left(\lim_{x\to\infty} \frac{1}{x}\right)$$
$$= 0 \cdot 0 = 0.$$

$$\lim_{x\to\infty} \frac{1-x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{x}{x^2}}{\frac{x^2}{x^2}} = \lim_{x\to\infty}\left(\frac{1}{x^2} - \frac{1}{x}\right) \stackrel{(2.)}{=} \lim_{x\to\infty} \frac{1}{x^2} - \lim_{x\to\infty} \frac{1}{x}$$
$$= 0 - 0 = 0.$$

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to\infty}\left(\frac{1}{x^{n-1}} \cdot \frac{1}{x}\right) \stackrel{(4.)}{=} \left(\lim_{x\to\infty} \frac{1}{x^{n-1}}\right)\left(\lim_{x\to\infty} \frac{1}{x}\right)$$
$$= 0 \cdot 0 = 0, \text{ for all } n > 1.$$

Limits of Rational Functions.

To compute the limit of a rational function at infinity:

- divide **both numerator and denominator** by the highest power of x **in the denominator**;
- then apply the limit laws ...

Example (From the Book, Section 2.6.)

Evaluate the limit of $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ as $x \to \infty$.

Soln. $\dfrac{(3x^2 - x - 2)/x^2}{(5x^2 + 4x + 1)/x^2} = \dfrac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + 4\frac{1}{x} + \frac{1}{x^2}} \to \dfrac{3 - 0 - 0}{5 + 0 + 0} = \dfrac{3}{5}$, as $x \to \infty$.

Example (MA161 Paper 1, 2012/2013.)

Evaluate $\displaystyle\lim_{x \to \infty} \frac{x^4 - 4x^2 + 4}{x^5 + 5x^3}$.

Solution:

$$\frac{(x^4 - 4x^2 + 4)/x^5}{(x^5 + 5x^3)/x^5} = \frac{\frac{1}{x} - 4\frac{1}{x^3} + 4\frac{1}{x^5}}{1 + 5\frac{1}{x^2}} \to \frac{0 + 0 + 0}{1 + 0} = \frac{0}{1} = 0, \text{ as } x \to \infty.$$

Squirrels Revisted.

Example.

A team of researchers studying red squirrels in Ireland has determined that the population density (i.e., the number of squirrels per square kilometre) in t years from now can be modelled as

$$P(t) = \frac{200t^2 + 50t + 900}{t^3 + 30}.$$

1. What is the current population density? $P(0) = \frac{900}{30} = 30$.
2. What will the population density be in 3 years time? $P(3) = \frac{200 \cdot 3^2 + 50 \cdot 3 + 900}{3^3 + 30} = 50$.
3. What can be predicted about the **long-term** population density? $\dfrac{(200t^2 + 50t + 900)/t^3}{(t^3 + 30)/t^3} \to 0$ as $t \to \infty$: over time, the red squirrels will disappear. ☹

Exercises.

1. For what values of k is the function $f(x)$ continuous at $x = 2$?

$$f(x) = \begin{cases} 3x^4 - 5x^3 + x + 3, & \text{if } x < 2, \\ kx^2 + 3x - 4, & \text{if } x \geqslant 2. \end{cases}$$

2. Evaluate the following limits.
   (i) $\lim_{x \to \infty} 2^x$.
   (ii) $\lim_{x \to -\infty} 2^x$.

3. Evaluate the following limits.
   (i) $\lim_{x \to \infty} \dfrac{2x - 9x^3}{2x + 3x^3}$.
   (ii) $\lim_{x \to \infty} \dfrac{x^5 - x^4 + x^3 + x^2}{x^4 + 23\,112}$.
   (iii) $\lim_{x \to \infty} \dfrac{12x - 18x^2 + 6}{8x + x^3 + 16}$.

4. New research on the red squirrel has called into question the population model used above. The team **now** believes the model should be

$$P(t) = \frac{200t^2 + 50t + 900}{t + 30}.$$

What is the predicted long-term population density?