Week 6: Limits and Continuity.

MA161/MA1161: Semester 1 Calculus.

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Recall: Limits.

[Section 2.2 of the Book]
The notion of a **limit** is one of the most fundamental in Calculus, and is the gateway to understanding the notion of a **derivative**. Here is the key idea:

The idea of a limit

We say that the **limit of a function f at the point p is L** if we can make $f(x)$ as close to L as we would like, by taking x as close to p as is necessary. We write this as

$$\lim_{x \to p} f(x) = L.$$

Example

The limit of $f(x) = \frac{|x|}{x}$ as $x \to 0$ **does not exist**, as $f(x) = -1$ for $x < 0$ and $f(x) = +1$ for $x > 0$. Thus, no matter how close to 0 you choose your x, there is no single value L that $f(x)$ comes close to.

More Examples.

Example

Suppose that $f(x)$ represents the distance (measured in meters) you have travelled at time x (measured in seconds).
To determine the **average speed** over the time from x_0 to x, we compute $\dfrac{f(x) - f(x_0)}{x - x_0}$.
To determine your **velocity**, i.e., the instantaneous change of distance, we need to compute $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.

Example

Sometimes we care about the long-term behaviour of a function. For example, if $p(t)$ describes the size of the population of bacteria at time t, we might want to know whether in the long term the population dies out, reaches a stable level, or just keeps growing and growing . . . Here we need to compute $\lim\limits_{t \to \infty} p(t)$.

How to Estimate a Limit.

Suppose that we want to estimate $\lim_{x \to p} f(x)$.

1. Make a **table of function values** $f(x)$ for values of x **near** p. Include points both to the left and to the right of p. (Avoid $x = p$.)
2. Check if these values tend to approach some fixed value.
3. If so (and if this value is the same on the left and on the right of p), then you can take this value as an estimate for the limit L. If not, the limit might not exist after all.

(Watch out for rounding errors when using a calculator ...)

Examples

Estimate the limit ...

1. ... of $f(x) = \dfrac{x^2 - 4}{x^3 - 3x - 2}$ as $x \to 2$.
2. ... of $f(x) = \dfrac{x}{x^2}$ as $x \to 0$.
3. ... of $f(x) = \dfrac{|x|}{x}$ as $x \to 0$.

One-Sided Limits.

The last example shows that estimates on the left of p can suggest a different limit than those on the right. This gives rise to the notion of **one-sided** limits.

Left-Hand Limit and Right-Hand Limit.

We write

$$\lim_{x \to p^-} f(x) = L \text{ (or } \lim_{x \to p^+} f(x) = L\text{)}$$

if $f(x)$ approaches L, as x approaches p from the left (or right). This means that we can make $f(x)$ as close to L as we would like by taking values of $x < p$ (or $x > p$) as close to p as necessary.

Example

$\lim_{x \to 0^-} \frac{|x|}{x} = -1$ and $\lim_{x \to 0^+} \frac{|x|}{x} = +1$.

2-Sided Limits.

The (two-sided) limit of $f(x)$ at p exists and is equal to L if and only if **both** one-sided limits of $f(x)$ at p exist and are equal to L.

$\lim_{x \to p} f(x) = L$ if and only if $\lim_{x \to p^-} f(x) = L$ and $\lim_{x \to p^+} f(x) = L$.

Example

Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2, & x \leqslant 1, \\ x, & x > 1. \end{cases}$$

What is its limit as $x \to 1$?
We have one-sided limits $\lim_{x \to 1^+} f(x) = 1$ and $\lim_{x \to 1^-} f(x) = 1$.
Hence the two-sided limit exists and $\lim_{x \to 1} = 1$.

More One-Sided Limits.

Example

Consider the piecewise defined function

$$f(x) = \begin{cases} x^2, & x \leqslant 1, \\ -x, & x > 1. \end{cases}$$

What is $\lim_{x \to 1} f(x)$?

Example

What is the limit of $f(x) = \dfrac{x}{\sqrt{x}}$ as $x \to 0$?

Infinite Limits.

Infinite Limits.

If a function $f(x)$ is defined at all points near $x = p$ then

$$\lim_{x \to p} f(x) = \infty$$

means that we can make $f(x)$ as large (and positive) as we would like by taking x as close to p as necessary. Similarly

$$\lim_{x \to p} f(x) = -\infty$$

means that we can make $f(x)$ as large in the negative direction as we would like by taking x as close to p as necessary.

Example

Find, if they exist, $\lim\limits_{x \to 0} \dfrac{1}{x}$ and $\lim\limits_{x \to 0} \dfrac{1}{x^2}$.

Calculating Limits.

What is

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}?$$

Estimating limits is one thing.
We prefer to calculate limits.
This yields precise results, and, in many cases, is less work.

Two types of limits are sort of obvious:

- $\lim_{x \to p} c = c$ for any constant $c \in \mathbb{R}$;
- and $\lim_{x \to p} x = p$.

Many limit problems can be reduced to these two, on the basis of the **Limit Laws**.

Limit Laws.

Limit Laws.

Suppose that $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are functions, that $p \in \mathbb{R}$ and that the limits $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ exist. Then:

1. $\lim_{x\to p} \big(f(x) + g(x)\big) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$.
2. $\lim_{x\to p} \big(f(x) - g(x)\big) = \lim_{x\to p} f(x) - \lim_{x\to p} g(x)$.
3. $\lim_{x\to p} \big(c\, f(x)\big) = c \lim_{x\to p} f(x)$ for all constants $c \in \mathbb{R}$.
4. $\lim_{x\to p} \big(f(x)g(x)\big) = \Big(\lim_{x\to p} f(x)\Big)\Big(\lim_{x\to p} g(x)\Big)$.
5. $\lim_{x\to p} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)}$, provided that $\lim_{x\to p} g(x) \neq 0$.

The limit of a sum is the sum of the limits . . . :
If $f(x)$ is close to L and $g(x)$ is close to M, then it is easy to believe that $f(x) + g(x)$ is close to $L + M$ . . .

For Example.

A consequence of Limit Law 4 is the **Power Law**:

- $\lim_{x\to p} f(x)^n = \left(\lim_{x\to p} f(x)\right)^n$; in particular, $\lim_{x\to p} x^n = p^n$.

By Limit Law 5,

$$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$$

provided that both limits exist and that $\lim_{x\to -2} 5-3x \neq 0$.

Using laws 1,2 and 3, and the Power Law from above, we find that $\lim_{x\to -2}(5-3x) = \lim_{x\to -2} 5 - \lim_{x\to -2} 3x = 5-3(-2) = 11$, and that

$$\begin{aligned}\lim_{x\to -2}(x^3+2x^2-1) &= \lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1 \\ &= (-2)^3 + 2(-2)^2 - 1 = -1,\end{aligned}$$

Hence, the limit is $-\frac{1}{11}$.

Direct Substitution.

Did you notice? In the last example, with $f(x) = \frac{x^3+2x^2-1}{5-3x}$, we found that $\lim_{x\to -2} f(x) = -\frac{1}{11} = f(-2)$: The limit of $f(x)$ as $x \to p$ is the same as the function value $f(x)$ at $x = p$! This is not an accident, it's an instance of a general phenomenon, known as "continuity", which we will discuss shortly. For now, we record the following.

Direct Substitution Property.

If f is a polynomial, or a rational function, and if p is a point in the domain of f, then

$$\lim_{x\to p} f(x) = f(p).$$

This allows us to calculate certain limits as function values, by substituting p for x in $f(x)$.

Examples.

The following can be useful when combined with direct substitution.

If $f(x) = g(x)$ for all $x \neq p$, then $\lim_{x \to p} f(x) = \lim_{x \to p} g(x)$, provided the limits exist.

Example

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} x + 1 = 2.$$

Example (MA161 Paper 1, 2012/13)

Evaluate the limit $\lim_{x \to -8} \frac{x^2 + 11x + 24}{x + 8}$.

Example (MA101 Summer Exam, 2012)

Evaluate the limit $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$.

Continuity.

We have seen that the limit of a function $f(x)$ as x approaches p can often be found by simply calculating the value of f at p. In such a situation we say that "f is **continuous** at p".

A function $f\colon D \to \mathbb{R}$ is **continuous** at a point $p \in D$ if

$$\lim_{x \to p} f(x) = f(p).$$

Note that this definition of continuity implicitly requires that

- $f(p)$ is defined (that is, p is in the domain of f),
- the limit $\lim_{x \to p} f(x)$ exists,
- and $\lim_{x \to p} f(x) = f(p)$.

The function f is continuous at p if $f(x)$ approaches $f(p)$ as x approaches p: small changes in x cause only small changes in $f(x)$. A continuous process is one that takes place gradually, without interruption or abrupt change.

Examples of Lack of Continuity.

- The function
$$f(x) = \frac{x^2 - x - 2}{x - 2}$$
is not continuous at $x = 2$, as $f(2)$ is not defined.
- The function
$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$
is not continuous at $x = 0$, as $\lim_{x \to 0} f(x)$ does not exist.
- The function
$$f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2}, & x \neq 2, \\ 1, & x = 2, \end{cases}$$
is not continuous at $x = 2$, as $\lim_{x \to 2} f(x) = 3 \neq 1 = f(x)$.

Continuity on an Interval.

One-Sided Continuity.

A function $f\colon D \to \mathbb{R}$ is continuous from the left (or right) at a point $p \in D$ if

$$\lim_{x \to p^-} f(x) = f(p) \quad \text{(or } \lim_{x \to p^+} f(x) = f(p)\text{).}$$

A function $f\colon D \to \mathbb{R}$ is **continuous on an interval** $I \subseteq D$ if it is continuous at every point p in the interval I.

Geometrically, you ocan think of a function that is continuous on an interval as a function whose graph can be drawn without lifting the pen from the paper.

Examples.

Example

Sketch the graph of the function

$$f(x) = \begin{cases} x+1, & x \leqslant 0, \\ e^x, & x > 0, \end{cases}$$

and determine if it is continuous at $x = 0$.

Example

Sketch the graph of the function

$$f(x) = \begin{cases} x-1, & x \leqslant 0, \\ e^x, & x > 0, \end{cases}$$

and determine if it is continuous at $x = 0$.

Exercises.

1. Evaluate the following limits
   (i) $\lim_{x\to 0} \frac{e^x+1}{x+1}$.
   (ii) $\lim_{x\to -1} \frac{x^2+x-2}{x-1}$.
   (iii) $\lim_{x\to 1} \frac{x^2+x-2}{x-1}$.
   (iv) $\lim_{x\to -3} \frac{x^2-x-12}{x+3}$.
   (v) $\lim_{x\to 5} \frac{x^3-3x^2-7x-15}{x-5}$.