Week 5: Logarithms; Limits.

MA161/MA1161: Semester 1 Calculus.

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Assignment 1 is online.

- The **Midterm 1** assignment is **now online** at

  http://mathswork.nuigalway.ie/webwork2/1920-MA161

  with a deadline of Tuesday, October 22.
- Visit one of the **Tutorials**:
  1. Monday at 18:00 in IT204,
  2. Tuesday at 11:00 in IT206,
  3. Tuesday at 16:00 in ADB-1020,
  4. Wednesday at 13:00 in CA101,
  5. Wednesday at 18:00 in AM110,
  6. Thursday at 13:00 in IT125G,

  and avail of the support in SUMS.

Composition of Functions.

It is easy to make **new functions from old** ones, by forming their **sum**, **difference** or **product**, and you've see many examples of this already:
A polynomial is a sum of multiples of power functions.
And a rational function is the **quotient** of two polynomials.
Another important operation of this type is called **composition**.

The **composite**, $f \circ g$, of two functions f and g is defined by the rule

$$(f \circ g)(x) = f(g(x)).$$

We say "f after g" for $f \circ g$, and pronounce $f(g(x))$ as "f of g of x". It means that we first apply g to x and then apply f to the result: f after g, for short.

Examples

1. Let $f(x) = x^2$ and $g(x) = \sin(x)$. Find $f \circ g$ and $g \circ f$.
2. Let $f(x) = 3x + 4$ and $g(x) = 2x^2 + 5x$. Find $f \circ g$ and $g \circ f$.

One-to-one Functions.

One-to-one Functions.

A function $f\colon D \to \mathbb{R}$ is called a **one-to-one function** if it never takes on the same value twice, that is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

Examples

- Is $f(x) = x^3$ one-to-one?
- Is $f(x) = x^2$ one-to-one?

- A function is one-to-one if it passes the **horizontal line test.**

Inverse Functions.

Suppose $f\colon D \to C$ is a one-to-one function with range C. Then its **inverse function** f^{-1} has **domain** C and **range** D and is defined by

$$f^{-1}(x) = y \Leftrightarrow f(y) = x, \text{ for any } x \in C.$$

Find the inverse function of $f(x) = x^3 + 2$.

Cancellation.

Suppose $f\colon D \to C$ is a one-to-one function with inverse f^{-1}. Then

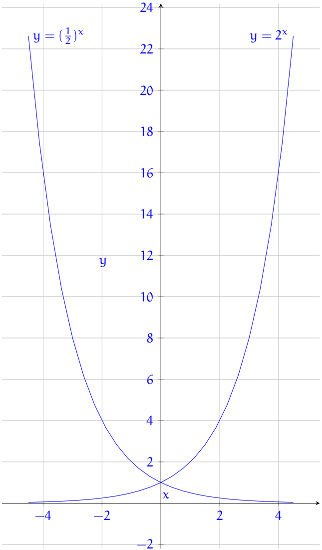
- $f^{-1}(f(x)) = x$ for every $x \in D$.
- $f(f^{-1}(x)) = x$ for every $x \in C$.

- **Do not confuse** the inverse function $f^{-1}(x)$ with the **reciprocal** $f(x)^{-1} = \frac{1}{f(x)}$.
- Not every function has an inverse.

Recall: Properties of The Exponential Function.

Some properties of $f(x) = a^x$:

- $f(0) = a^0 = 1$.
- if $x = n \in \mathbb{N}$ then $f(x) = a^n = a \cdot a \cdots a$ (n factors).
- $f(-x) = a^{-x} = 1/a^x$.
- $f(1/x) = a^{1/x} = \sqrt[x]{a}$.
- If $x = p/q \in \mathbb{Q}$ then $f(x) = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.



Laws of Exponents

1. $a^{x+y} = a^x a^y$.
2. $a^{x-y} = \dfrac{a^x}{a^y}$.
3. $a^{xy} = (a^x)^y$.
4. $(ab)^x = a^x b^x$.

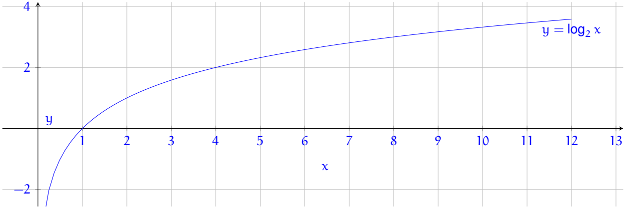
Logarithmic Functions.

see Section 1.6 of the Book . . .

- Recall: The **logarithmic function** $\log_a$, with **base** a, is the **inverse** of the **exponential function** a^x, with **base** a:

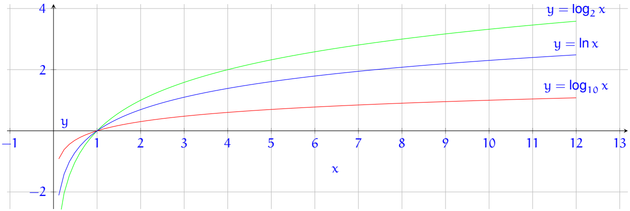
$$y = \log_a x \Leftrightarrow a^y = x.$$

- Consider the case $a = 2$.



- $\log_2 x$ is the inverse of $f(x) = 2^x$; e.g., $\log_2(8) = 3$ since $2^3 = 8$.
- $\log_{10}(x)$ is the power into which to raise 10 in order to get x; e.g., $\log_{10}(1\,000\,000) = 6$.
- For $n \in \mathbb{N}$, the number $\log_{10}(n)$ is, roughly, the number of (decimal) digits of n.

Properties of Logarithmic Functions.



Logarithms and exponential functions are inverse to each other:

- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$.
- $a^{\log_a(x)} = x$ for all $x > 0$.

Logarithm Laws

- $\log_a(xy) = \log_a(x) + \log_a(y)$.
- $\log_a(x/y) = \log_a(x) - \log_a(y)$.
- $\log_a(x^r) = r \log_a(x)$.

Logarithms: Common Bases.

The most common bases for logarithms are:

- base $a = 10$, relates to decimal representation of numbers; $\log_{10} 10 = 1, \quad \log_{10} 1000 = 3, \quad \log_{10} 10^n = n$.
- base $a = 2$, relates to binary representation of numbers: $64 = 2^6 = (1\,000\,000)_2 \implies \log_2 64 = 6$.
- base $e = 2.71828182845905\ldots$, Euler's number. The **natural logarithm** is written as

$$\ln x := \log_e x.$$

To **convert** from one base, b say, to another, a, use

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} = \frac{\ln x}{\ln a}.$$

$\log_3(1000) = \log_{10}(1000)/\log_{10}(3) = 3/0.4771 = 6.2877$.

Applications of Logarithmic Functions.

1. (Logarithmic Scales.) If a function is given by the equation $f(x) = x^n$ for some n, then $\log_a(f(x)) = n\log_a(x)$, i.e., the graph of $\log_a(f(x))$ against $\log_a(x)$ is a straight line, with **slope** n.
2. (Bacterial Growth revisted.) Recall from earlier, that a population of Cyanobacteria can double four times every day. If at time $t = 0$ the population is $p(0) = 1,000$ cells, the population at time t, measured in hours, can be modelled as
$$p(t) = 1000 \times 2^{t/6}.$$
How many hours does it take for the population to reach $250,000$?
3. (Stewart, Sec 1.6, Q.57) A bacteria population starts with 100 bacteria and doubles every three hours. So the number of bacteria after t hours can be modelled as $f(t) = 100 \times 2^{t/3}$. When will the population reach $50,000$?

Limits.

see Section 2.1 of the Book . . .

The notion of a **limit** is one of the most fundamental in Calculus, and is the gateway to understanding the notion of a **derivative**. Here is the key idea:

The idea of a limit

We say that the **limit of a function f at the point p is L** if we can make $f(x)$ as close to L as we would like, by taking x as close to p as is necessary. We write this as

$$\lim_{x \to p} f(x) = L.$$

This description raises a few questions:

1. What does it mean to "take x as close to p as necessary"?
2. What does it mean to "make $f(x)$ as close to L as we would like"?

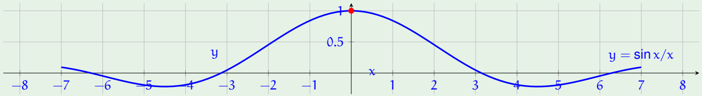
Let's look at how this works in a few concrete examples.

The Trigonometric Limit, For Example

- The limit of $f(x)$ as $x \to p$, if it exists, is determined by function values $f(x)$ for values of x **close to** p but **different from** p.
- Sometimes, $f(p)$ is not even defined, as p is not in the domain of f. Still, the limit of $f(x)$ as $x \to p$ might exist.

Example

Consider the function $f(x) = \dfrac{\sin x}{x}$ and the limit of $f(x)$ as $x \to 0$. We cannot compute $f(0)$ as 0 is not in the domain of f. But we can compute the value of $f(x)$ for any x as close to 0 as we like. Plotting the points obtained in this way, yields the graph of $f(x)$:



This suggests that $\lim_{x\to 0} f(x) = 1$.

More Examples.

Example

Suppose that $f(x)$ represents the distance (measured in meters) you have travelled at time x (measured in seconds).
To determine the **average speed** over the time from x_0 to x, we compute $\frac{f(x) - f(x_0)}{x - x_0}$.
To determine your **velocity**, i.e., the instantaneous change of distance, we need to compute $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

Example

Sometimes we care about the long-term behaviour of a function. For example, if $p(t)$ describes the size of the population of bacteria at time t, we might want to know whether in the long term the population dies out, reaches a stable level, or just keeps growing and growing . . . Here we need to compute $\lim_{t \to \infty} p(t)$.

How to Estimate a Limit.

Suppose that we want to estimate $\lim_{x \to p} f(x)$.

1. Make a **table of function values** $f(x)$ for values of x **near** p. Include points both to the left and to the right of p. (Avoid $x = p$.)
2. Check if these values tend to approach some fixed value.
3. If so (and if this value is the same on the left and on the right of p), then you can take this value as an estimate for the limit L. If not, the limit might not exist after all.

(Watch out for rounding errors when using a calculator ...)

Examples

Estimate the limit ...

1. ... of $f(x) = \dfrac{x^2 - 4}{x^3 - 3x - 2}$ as $x \to 2$.
2. ... of $f(x) = \dfrac{x}{x^2}$ as $x \to 0$.
3. ... of $f(x) = \dfrac{|x|}{x}$ as $x \to 0$.

One-Sided Limits.

The last example shows that estimates on the left of p can suggest a different limit than those on the right. This gives rise to the notion of **one-sided** limits.

Left-Hand Limit and Right-Hand Limit.

We write

$$\lim_{x \to p^-} f(x) = L \text{ (or } \lim_{x \to p^+} f(x) = L\text{)}$$

if $f(x)$ approaches L, as x approaches p from the left (or right). This means that we can make $f(x)$ as close to L as we would like by taking values of $x < p$ (or $x > p$) as close to p as necessary.

Example

$\lim_{x \to 0^-} \frac{|x|}{x} = -1$ and $\lim_{x \to 0^+} \frac{|x|}{x} = +1$.

2-Sided Limits.

The (two-sided) limit of $f(x)$ at p exists and is equal to L if and only if **both** one-sided limits of $f(x)$ at p exist and are equal to L.

$\lim_{x \to p} f(x) = L$ if and only if $\lim_{x \to p^-} f(x) = L$ and $\lim_{x \to p^+} f(x) = L$.

Example

Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2, & x \leqslant 1, \\ x, & x > 1. \end{cases}$$

What is its limit as $x \to 1$?

Exercises.

1. Simplify the following expressions.
   (i) $e^{3\ln 4}$.
   (ii) $\log_4 64 + \log_2 1024$.
   (iii) $\ln(2e^{-x/2}) - \ln 2 + \frac{x}{2}$.
   (iv) $\ln 81/\ln 3$.
2. Write down the values of

$$f(x) = \frac{\cos x}{x - \pi/2}$$

for some values of x near $\pi/2$. Then guess $\lim_{x\to\pi/2} f(x)$.

3. Estimate the value of

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}.$$

4. The graph of

$$f(x) = \begin{cases} 1+x, & x < -1, \\ \frac{1}{2}x^2, & x \geqslant -1, \end{cases}$$

is shown below. Calculate the following limits:
(i) $\lim_{x\to 0^+} f(x)$, (ii) $\lim_{x\to 0^-} f(x)$,
(iii) $\lim_{x\to 0} f(x)$, (iv) $\lim_{x\to 1^+} f(x)$,
(v) $\lim_{x\to 1^-} f(x)$, (vi) $\lim_{x\to 1} f(x)$.