Week 4. A Catalog of Functions.

MA161/MA1161: Semester 1 Calculus.

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Reminder: Assignments.

Problem Set 1 is due on Friday, October 7 at 5pm (sharp).
Ideally, submit your answers before Friday, September 30.
There will be no extension of the deadline.
If you need assistance, avail of the support in SUMS, and in the tutorials:

1. Monday at 18:00 in IT202,
2. Tuesday at 11:00 in IT206,
3. Tuesday at 18:00 in AC201,
4. Wednesday at 14:00 in AC214,
5. Wednesday at 18:00 in AM104,
6. Thursday at 13:00 in IT125G.

A Catalog of Functions.

[Following Section 1.2 of the Book]

We'll now spend some time reviewing the most common **essential functions**. These include:

1. Linear Functions;
2. Polynomials;
3. Power Functions;
4. Rational Functions;
5. Algebraic Functions;
6. Trigonometric Functions;
7. Exponential Functions;
8. Logarithms.

1. Linear Functions.

A **linear function** is one whose graph is a straight line. It can be represented by a formula of the form

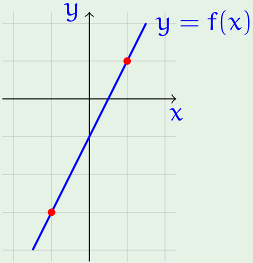
$$f(x) = mx + b,$$

where m is the **slope**, and b is the y**-intercept**.

Example

$f(x) = 2x - 1$.

Slope: 2.
y-intercept: -1.
$f(1) = 1$, $f(-1) = -3$, …



A linear function with slope $m = 0$ is called a **constant** function.

2. Polynomials.

Definition

A function $P\colon D \to \mathbb{R}$ is called a **polynomial function** (or simply a polynomial) if it has the form

$$P(x) = a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_2\, x^2 + a_1\, x + a_0,$$

for some nonnegative integer n, and **coefficients** $a_0, a_1, \ldots, a_n \in \mathbb{R}$. If $a_n \neq 0$ the we say that P has **degree** n.

- The domain of any polynomial is $\mathbb{R}$.

- $f(x) = 1 = x^0$, $f(x) = x = x^1$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, ...
- $f(x) = 3x^3 + 2x^2 + x$, $\quad f(x) = \frac{1}{2} - x^3 + 1001x^5$.
- $f(x) = x^3$ has degree 3; also $f(x) = 1 - x^3 + 2x$ has degree 3.
- $f(x) = (x-1)(x-2)(x-3)$ is a polynomial of degree 3.
- A non-constant linear function is a polynomial of degree 1 ...

Quadratics and Cubics.

The most important examples of polynomials are the quadratics and the cubics.

- A **quadratic** is a polynomial of degree 2.

Examples

- $P(x) = x^2 - 1; \quad P(x) = 1 - 10x^2 + 4x; \quad \dots$
- $P(x) = K(x-a)(x-b)$ for any numbers $a, b, K \in \mathbb{R}$, $K \neq 0$.

- A **cubic** is a polynomial of degree 3.

Examples

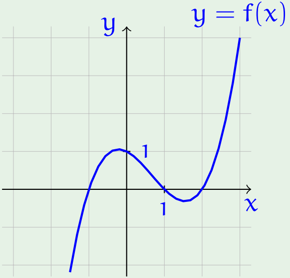
- $P(x) = x^3 - 1; \quad P(x) = 1 + 2x - 3x^2 + 45x^3; \quad \dots$
- $P(x) = K(x-a)(x-b)(x-c)$ for any numbers $a, b, K \in \mathbb{R}$, $K \neq 0$.

- Not every cubic has the form $P(x) = K(x-a)(x-b)(x-c)$ ...

The Graph of a Cubic.

Example

The following graph is that of a cubic polynomial. What is its equation?



- Assume P has an equation of the form

$$P(x) = K(x-a)(x-b)(x-c).$$

Find a, b, c, and K!

- **Solution:** a, b, and c are the zeros of P, hence

$$a = -1;\ b = 1;\ c = 2.$$

- $P(x) = K(x+1)(x-1)(x-2)$.
- Also $P(0) = 1$ in the graph.
- $P(0) = K(1)(-1)(-2) = 2K$.
- It follows that $K = \frac{1}{2}$

- **Answer:** $P(x) = \frac{1}{2}(x+1)(x-1)(x-2) = \frac{1}{2}x^3 - x^2 - \frac{1}{2}x + 1$.

3. Power Functions.

Definition

A **power function** is a function of the form $f(x) = x^a$ for some constant $a \in \mathbb{R}$. We call a the **exponent** of the function f.

Special cases:

(a) $a = n \in \mathbb{N}$:
f is a **polynomial**, like x^2, x^3, ...

(b) $a = \frac{1}{n}$ for some $n \in \mathbb{N}$:
f is a **root function**, like $\sqrt{x}$, $\sqrt[3]{x}$, ...

(c) $a = -1$:
f is the **reciprocal** function, $f(x) = \frac{1}{x}$.

Examples

$2^{-3} = (2^{-1})^3 = (\frac{1}{2})^3$; $\quad (\frac{1}{3})^{-4} = ((\frac{1}{3})^{-1})^4 = 3^4$.

4. Rational Functions.

Definition

A **rational function** f is a quotient of polynomials,

$$f(x) = \frac{P(x)}{Q(x)},$$

where both P and Q are polynomials.

The domain of the rational function $f(x) = P(x)/Q(x)$ consists of all points $x \in \mathbb{R}$ such that $Q(x) \neq 0$.

Example

The function $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$ is a rational function with domain

$$\{x \in \mathbb{R} : x \neq \pm 2\} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

5. Algebraic Functions.

Definition

A function f is called an **algebraic function** if it can be constructed from polynomials using the **algebraic operations** of addition, subtraction, multiplication, division, and taking roots.

Power functions and rational functions are algebraic functions.

Examples

$$f(x) = \sqrt{x^2+1}; \quad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x-1)\sqrt[3]{x+1}.$$

In **relativity theory**, the **mass** of a particle with velocity v is given by the algebraic function

$$f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

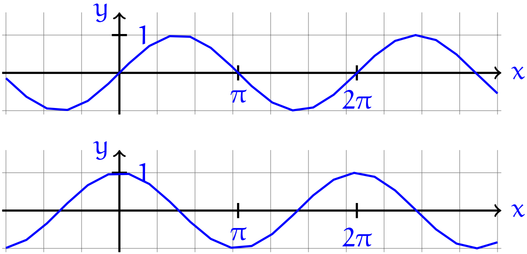
where m_0 is the mass of the particle at rest, and c is the speed of light.

6. Trigonometric functions.

- The most important **trigonometric functions** are:

  $\sin(x)$; $\cos(x)$; $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

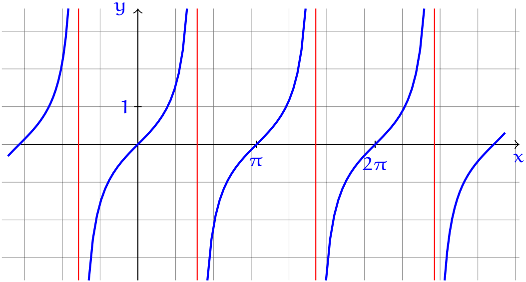
- In calculus, **angles** are measured in **radians**.
- The graphs of $\sin(x)$ and $\cos(x)$ look like:



- The domain of both $\sin(x)$ and $\cos(x)$ is $\mathbb{R}$, the range is the closed interval $[-1, 1]$.
- They are **periodic**:
  $f(x + 2\pi) = f(x)$ when $f(x) = \sin(x)$ or $f(x) = \cos(x)$.

The Tangent Function.

- $\sin(x) = 0$ when $x = n\pi$ for $n \in \mathbb{Z}$; e.g., $x = \pi$, $x = 2\pi$, etc.
- $\cos(x) = 0$ when $x = -\pi/2$, $x = \pi/2$, $x = 3\pi/2$, ...
- $\tan(x) = \sin(x)/\cos(x)$ is undefined whenever $\cos(x) = 0$.
- The graph of $\tan(x)$ looks like:



- Note that $\tan(x)$ has **period** π: $\tan(x + \pi) = \tan(x)$.

7. Exponential Functions.

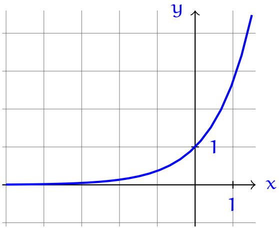
Definition

An **exponential function** is a function of the form

$$f(x) = a^x$$

for some **positive** constant **base** $a > 0$.

- The graph of $f(x) = e^x$ (where $e = 2.718281\ldots$) looks like:

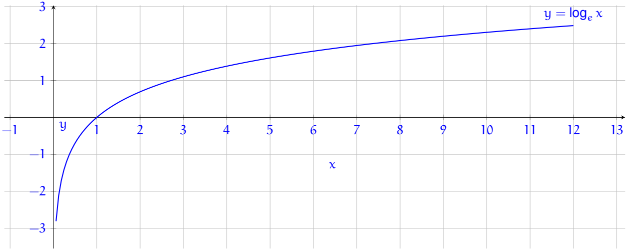


- The **domain** of e^x is all of $\mathbb{R}$, the **range** is $(0, \infty)$.
- Exponential functions are useful for modeling natural phenomena like population growth and radioactive decay, . . .

8. Logarithmic Functions.

The **logarithmic functions** $f(x) = \log_a(x)$, where the **base** a is a positive constant, are the inverse functions of the exponential functions: $y = \log_a(x)$ if $x = a^y$.

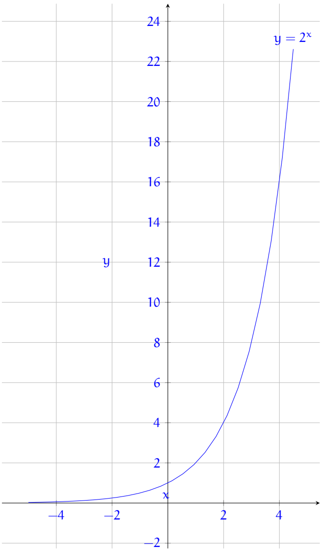
- The domain of $\log_a(x)$ is $(0, \infty)$, the range is $\mathbb{R}$.
- The graph of the **natural logarithm** $\ln(x) = \log_e(x)$ looks like:



Exponential Functions.

[see Section 1.5 of the Book]

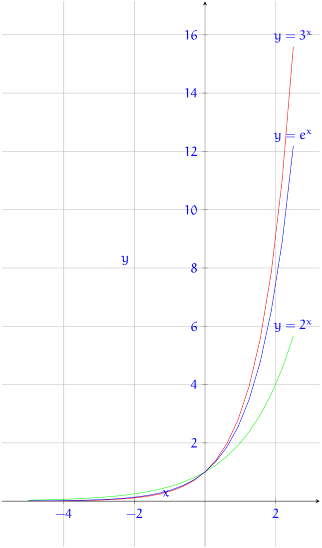
- The function $f(x) = 2^x$ is called and **exponential** function, because the variable, x, is the **exponent**.
- It should not be confused with the power function $f(x) = x^2$, where the variable is in the **base**.
- One characteristic of an exponential function like $f(x) = 2^x$ is that $f(x)$ **grows very rapidly** as x increases.
- It can thus be used to model situations, where things are **doubling all the time**: 1, 2, 4, 8, 16, 32, 64, 128, ...



Properties of The Exponential Function $f(x) = a^x$.

Some properties of $f(x) = a^x$:

- $f(0) = a^0 = 1$.
- if $x = n \in \mathbb{N}$ then $f(x) = a^n = a \cdot a \cdots a$ (n factors).
- $f(-x) = a^{-x} = 1/a^x$.
- $f(1/x) = a^{1/x} = \sqrt[x]{a}$.
- If $x = p/q \in \mathbb{Q}$ then $f(x) = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.



Laws of Exponents

1. $a^{x+y} = a^x a^y$.
2. $a^{x-y} = \dfrac{a^x}{a^y}$.
3. $a^{xy} = (a^x)^y$.
4. $(ab)^x = a^x b^x$.

Applications of Exponential Functions.

1. Bacterial Growth

(see `http://wikipedia.org/wiki/Bacterial_growth`)
During the so-called "log phase" of bacterial growth in batch culture, in a controlled environment a Cyanobacteria population will double in size four times every day. If at time $t = 0$ the population is $p(0) = 1,000$ cells, show that the population at time t, measured in hours, can be modelled as

$$p(t) = 1000 \times 2^{t/6}.$$

2. Radioactive Decay

Suppose that a radioactive substance has **half-life** H. This means that, compared to time t, only half of the substance remains at time $t + H$. Show that this can be described by the function

$$f(t) = C \times (\tfrac{1}{2})^{t/H}.$$

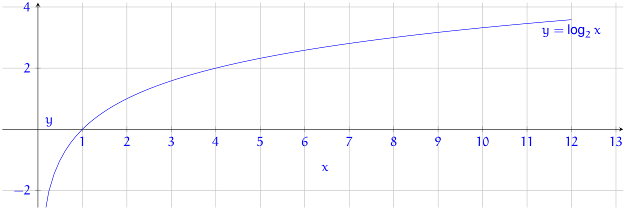
Logarithmic Functions.

[see Section 1.6 of the Book]

- Recall: The **logarithmic function** $\log_a$, with **base** a, is the **inverse** of the **exponential function** a^x, with **base** a:

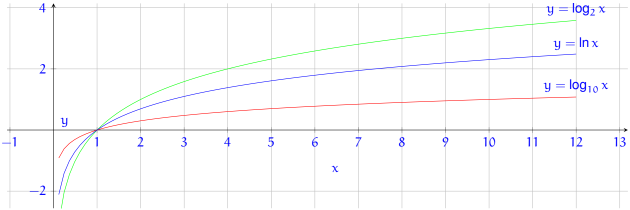
$$y = \log_a x \Leftrightarrow a^y = x.$$

- Consider the case $a = 2$.



- $\log_2 x$ is the inverse of $f(x) = 2^x$; e.g., $\log_2(8) = 3$ since $2^3 = 8$.
- $\log_2(x)$ is the power into which to raise 2 in order to get x.
- $\log_{10}(x)$ is the power into which to raise 10 in order to get x; e.g., $\log_{10}(1\,000\,000) = 6$.
- For $n \in \mathbb{N}$, the number $\log_{10}(n)$ is, roughly, the number of (decimal) digits of n.

Properties of Logarithmic Functions.



Logarithms and exponential functions are inverse to each other:

- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$.
- $a^{\log_a(x)} = x$ for all $x > 0$.

Logarithm Laws

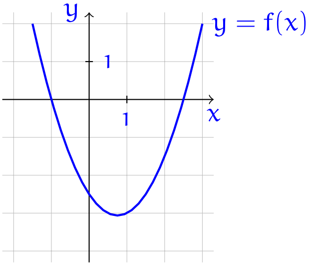
- $\log_a(xy) = \log_a(x) + \log_a(y)$.
- $\log_a(x/y) = \log_a(x) + \log_a(y)$.
- $\log_a(x^r) = r \log_a(x)$.

Applications of Logarithmic Functions.

1. (Logarithmic Scales.) If a function is given by the equation $f(x) = x^n$ for some n, then $\log_a(f(x)) = n\log_a(x)$, i.e., the graph of $\log_a(f(x))$ against $\log_a(x)$ is a straight line, with **slope** n.
2. (Bacterial Growth revisted.) Recall from earlier, that a population of Cyanobacteria can double four times every day. If at time $t = 0$ the population is $p(0) = 1,000$ cells, the population at time t, measured in hours, can be modelled as
$$p(t) = 1000 \times 2^{t/6}.$$
How many hours does it take for the population to reach $250,000$?
3. (Stewart, Sec 1.6, Q.57) A bacteria population starts with 100 bacteria and doubles every three hours. So the number of bacteria after t hours can be modelled as $f(t) = 100 \times 2^{t/3}$. When will the population reach $50,000$?

Exercises.

1. Here is the graph of a quadratic polynomial. What is its equation?

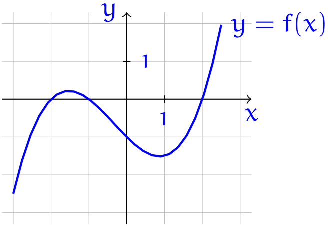


2. Find (a) $\log_{10} 1\,000\,000$, (b) $\log_2 1024$, (c) $\log_3 6561$.

3. The following graph is that of a cubic polynomial with equation

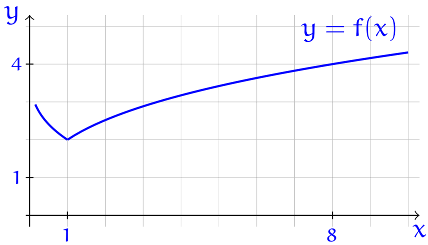
$$f(x) = K(x-a)(x-b)(x-c).$$

Find a, b, c, and K.



Exercises.

4. Here is the graph of the function $f(x) = 2x^{1/n}$, where n is a natural number. What is the value of n?



5. (MA160/MA161 Paper 1, 2012/13) A biologist estimates that there are currently 500 zebra mussels in Lough Corrib, and that this number is doubling every year.
   (i) Show that the population can be modelled as $P(t) = 500 \times 2^t$, where t is time in years, and $t = 0$ represents the current time.
   (ii) Find P_0 and k such that the formula can be expressed as $P(t) = P_0 e^{kt}$.
6. Find, correct to 3 decimal places, (a) $\log_2 15$, (b) $\log_2 56.25$, (c) $\log_3 16$.