Week 4: A Catalog of Functions.

MA161/MA1161: Semester 1 Calculus.

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Reminder: Tutorials.

There is a **3rd sheet with problems** to work on for download at

`http://schmidt.nuigalway.ie/ma161-1`

Tutorials:

1. Monday at 18:00 in IT204,
2. Tuesday at 11:00 in IT206,
3. Tuesday at 16:00 in ADB-1020,
4. Wednesday at 13:00 in CA101,
5. Wednesday at 18:00 in AM110,
6. Thursday at 13:00 in IT125G.

If you need further assistance, avail of the support in **SUMS**.

A Catalog of Functions.

see Section 1.2 of the Book . . .

We'll now spend some time reviewing the most common **essential functions**. These include:

1. Linear Functions;
2. Polynomials;
3. Power Functions;
4. Rational Functions;
5. Algebraic Functions;
6. Trigonometric Functions;
7. Exponential Functions;
8. Logarithms.

1. Linear Functions.

A **linear function** is one whose graph is a straight line. It can be represented by a formula of the form

$$f(x) = mx + b,$$

where m is the **slope**, and b is the y**-intercept**.

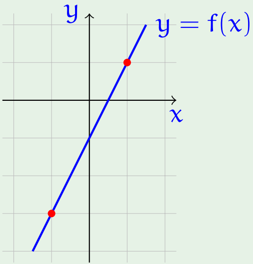
Example

$$f(x) = 2x - 1.$$

Slope: 2.

y-intercept: -1.

Some values:
$f(1) = 1$, $f(-1) = -3$, ...



A linear function with slope $m = 0$ is called a **constant** function.

2. Polynomials.

Definition

A function $P\colon D \to \mathbb{R}$ is called a **polynomial function** (or simply a **polynomial**) if it has the form

$$P(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_2\,x^2 + a_1\,x + a_0,$$

for some nonnegative integer n, and **coefficients** $a_0, a_1, \ldots, a_n \in \mathbb{R}$. If $a_n \neq 0$ the we say that P has **degree** n.

- The domain of any polynomial is $\mathbb{R}$.

- $f(x) = 1 = x^0$, $f(x) = x = x^1$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, …
- $f(x) = 3x^3 + 2x^2 + x$, $\quad f(x) = \frac{1}{2} - x^3 + 1001x^5$.
- $f(x) = x^3$ has degree 3; also $f(x) = 1 - x^3 + 2x$ has degree 3.
- $f(x) = (x-1)(x-2)(x-3)$ is a polynomial of degree 3.
- A **linear function** (with slope $m \neq 0$) is a polynomial of degree 1.
- …

Quadratics and Cubics.

The most important examples of polynomials are the quadratics and the cubics.

- A **quadratic** is a polynomial of degree 2.

Examples

- $P(x) = x^2 - 1; \quad P(x) = 1 - 10x^2 + 4x; \quad \ldots$
- $P(x) = K(x-a)(x-b)$ for any numbers $a, b, K \in \mathbb{R}$, $K \neq 0$.

- A **cubic** is a polynomial of degree 3.

Examples

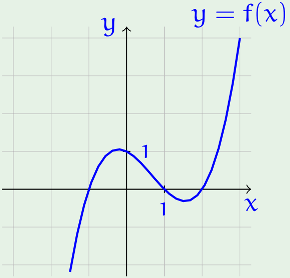
- $P(x) = x^3 - 1; \quad P(x) = 1 + 2x - 3x^2 + 45x^3; \quad \ldots$
- $P(x) = K(x-a)(x-b)(x-c)$ for any numbers $a, b, K \in \mathbb{R}$, $K \neq 0$.

- Not every cubic has the form $P(x) = K(x-a)(x-b)(x-c)$ ...

The Graph of a Cubic.

Example

The following graph is that of a cubic polynomial. What is its equation?



- Assume P has an equation of the form

$$P(x) = K(x-a)(x-b)(x-c).$$

Find a, b, c, and K!

- **Solution:** a, b, and c are the zeros of P, hence

$$a = -1;\ b = 1;\ c = 2.$$

- $P(x) = K(x+1)(x-1)(x-2)$.
- Also $P(0) = 1$ in the graph.
- $P(0) = K(1)(-1)(-2) = 2K$.
- It follows that $K = \frac{1}{2}$

- **Answer:** $P(x) = \frac{1}{2}(x+1)(x-1)(x-2) = \frac{1}{2}x^3 - x^2 - \frac{1}{2}x + 1$.

3. Power Functions.

Definition

A **power function** is a function of the form $f(x) = x^a$ for some constant $a \in \mathbb{R}$. We call a the **exponent** of the function f.

Special cases:

(a) $a = n \in \mathbb{N}$:
f is a **polynomial**, like x^2, x^3, ...

(b) $a = \frac{1}{n}$ for some $n \in \mathbb{N}$:
f is a **root function**, like $\sqrt{x}$, $\sqrt[3]{x}$, ...

(c) $a = -1$:
f is the **reciprocal** function, $f(x) = \frac{1}{x}$.

Examples

$2^{-3} = (2^{-1})^3 = (\frac{1}{2})^3$; $\quad (\frac{1}{3})^{-4} = ((\frac{1}{3})^{-1})^4 = 3^4$.

4. Rational Functions.

Definition

A **rational function** f is a quotient of polynomials,

$$f(x) = \frac{P(x)}{Q(x)},$$

where both P and Q are polynomials.

The domain of the rational function $f(x) = P(x)/Q(x)$ consists of all points $x \in \mathbb{R}$ such that $Q(x) \neq 0$.

Example

The function $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$ is a rational function with domain

$$\{x \in \mathbb{R} : x \neq \pm 2\} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

5. Algebraic Functions.

Definition

A function f is called an **algebraic function** if it can be constructed from polynomials using the **algebraic operations** of addition, subtraction, multiplication, division, and taking roots.

Power functions and rational functions are algebraic functions.

Examples

$$f(x) = \sqrt{x^2 + 1}; \quad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 1)\sqrt[3]{x + 1}.$$

In **relativity theory**, the **mass** of a particle with velocity v is given by the algebraic function

$$f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

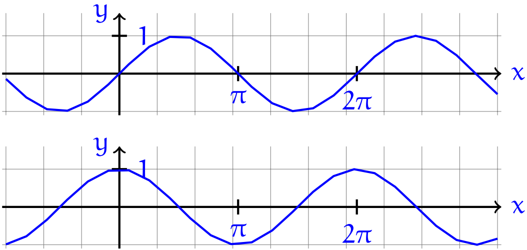
where m_0 is the mass of the particle at rest, and c is the speed of light.

6. Trigonometric functions.

- The most important **trigonometric functions** are:

  $\sin(x); \quad \cos(x); \quad \tan(x) = \frac{\sin(x)}{\cos(x)}.$

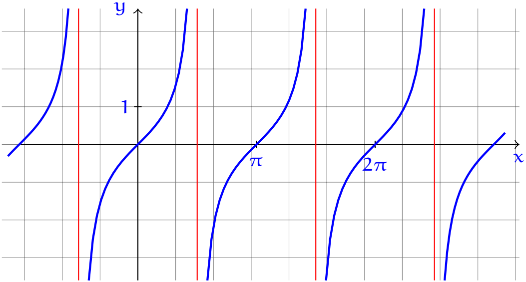
- In calculus, **angles** are measured in **radians**.
- The graphs of $\sin(x)$ and $\cos(x)$ look like:



- The domain of both $\sin(x)$ and $\cos(x)$ is $\mathbb{R}$, the range is the closed interval $[-1, 1]$.
- They are **periodic**:
  $f(x + 2\pi) = f(x)$ when $f(x) = \sin(x)$ or $f(x) = \cos(x)$.

The Tangent Function.

- $\sin(x) = 0$ when $x = n\pi$ for $n \in \mathbb{Z}$; e.g., $x = \pi$, $x = 2\pi$, etc.
- $\cos(x) = 0$ when $x = -\pi/2$, $x = \pi/2$, $x = 3\pi/2$, ...
- $\tan(x) = \sin(x)/\cos(x)$ is undefined whenever $\cos(x) = 0$.
- The graph of $\tan(x)$ looks like:



- Note that $\tan(x)$ has **period** π: $\tan(x + \pi) = \tan(x)$.

7. Exponential Functions.

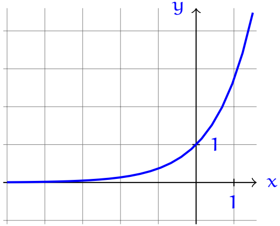
Definition

An **exponential function** is a function of the form

$$f(x) = a^x$$

for some **positive** constant **base** $a > 0$.

- The graph of $f(x) = e^x$ (where $e = 2.718281\ldots$) looks like:

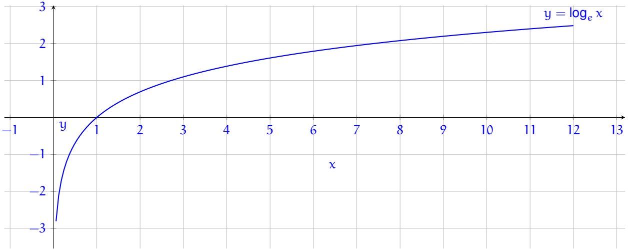


- The **domain** of e^x is all of $\mathbb{R}$, the **range** is $(0, \infty)$.
- Exponential functions are useful for modeling natural phenomena like population growth and radioactive decay, . . .

8. Logarithmic Functions.

The **logarithmic functions** $f(x) = \log_a(x)$, where the **base** a is a positive constant, are the inverse functions of the exponential functions: $y = \log_a(x)$ if $x = a^y$.

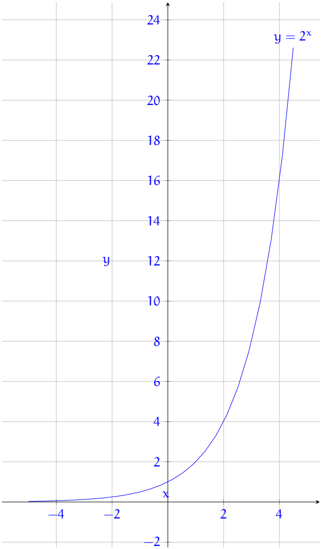
- The domain of $\log_a(x)$ is $(0, \infty)$, the range is $\mathbb{R}$.
- The graph of the **natural logarithm** $\ln(x) = \log_e(x)$ looks like:



Exponential Functions.

see Section 1.5 of the Book . . .

- The function $f(x) = 2^x$ is called an **exponential** function, because the variable, x, is the **exponent**.
- **Not to be confused** with the power function $f(x) = x^2$, where x is in the **base**.
- One characteristic of an exponential function like $f(x) = 2^x$ is that $f(x)$ **grows very rapidly** as x increases.
- It can thus be used to model situations, where things are **doubling all the time**: $1, 2, 4, 8, 16, 32, 64, 128, \ldots$



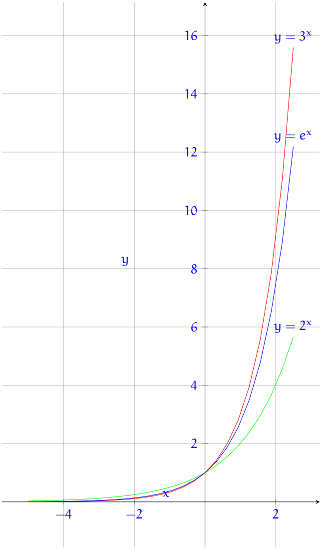
Properties of The Exponential Function $f(x) = a^x$.

Some properties of $f(x) = a^x$:

- $f(0) = a^0 = 1$.
- if $x = n \in \mathbb{N}$ then $f(x) = a^n = a \cdot a \cdots a$ (n factors).
- $f(-x) = a^{-x} = 1/a^x$.
- $f(1/x) = a^{1/x} = \sqrt[x]{a}$.
- If $x = p/q \in \mathbb{Q}$ then $f(x) = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

Laws of Exponents

1. $a^{x+y} = a^x a^y$.
2. $a^{x-y} = \dfrac{a^x}{a^y}$.
3. $a^{xy} = (a^x)^y$.
4. $(ab)^x = a^x b^x$.



Applications of Exponential Functions.

1. Bacterial Growth

(see http://wikipedia.org/wiki/Bacterial_growth)
During the so-called "log phase" of bacterial growth in batch culture, in a controlled environment a Cyanobacteria population will double in size four times every day. If at time $t = 0$ the population is $p(0) = 1,000$ cells, show that the population at time t, measured in hours, can be modelled as

$$p(t) = 1000 \times 2^{t/6}.$$

2. Radioactive Decay

Suppose that a radioactive substance has **half-life** H. This means that, compared to time t, only half of the substance remains at time $t + H$. Show that this can be described by the function

$$f(t) = C \times (\tfrac{1}{2})^{t/H}.$$

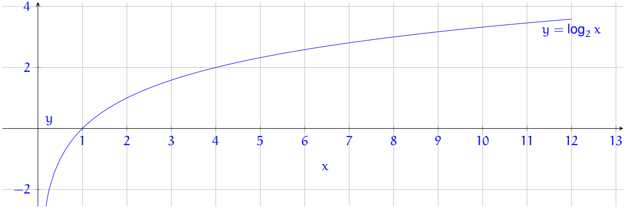
Logarithmic Functions.

see Section 1.6 of the Book . . .

- Recall: The **logarithmic function** $\log_a$, with **base** a, is the **inverse** of the **exponential function** a^x, with **base** a:

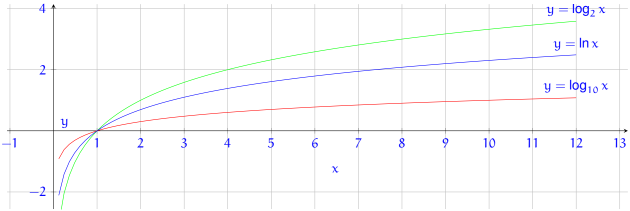
$$y = \log_a x \Leftrightarrow a^y = x.$$

- Consider the case $a = 2$.



- $\log_2 x$ is the inverse of $f(x) = 2^x$; e.g., $\log_2(8) = 3$ since $2^3 = 8$.
- $\log_{10}(x)$ is the power into which to raise 10 in order to get x; e.g., $\log_{10}(1\,000\,000) = 6$.
- For $n \in \mathbb{N}$, the number $\log_{10}(n)$ is, roughly, the number of (decimal) digits of n.

Properties of Logarithmic Functions.



Logarithms and exponential functions are inverse to each other:

- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$.
- $a^{\log_a(x)} = x$ for all $x > 0$.

Logarithm Laws

- $\log_a(xy) = \log_a(x) + \log_a(y)$.
- $\log_a(x/y) = \log_a(x) + \log_a(y)$.
- $\log_a(x^r) = r \log_a(x)$.

Applications of Logarithmic Functions.

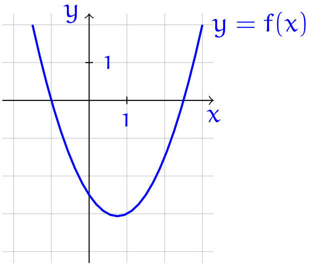
1. (Logarithmic Scales.) If a function is given by the equation $f(x) = x^n$ for some n, then $\log_a(f(x)) = n \log_a(x)$, i.e., the graph of $\log_a(f(x))$ against $\log_a(x)$ is a straight line, with **slope** n.
2. (Bacterial Growth revisted.) Recall from earlier, that a population of Cyanobacteria can double four times every day. If at time $t = 0$ the population is $p(0) = 1,000$ cells, the population at time t, measured in hours, can be modelled as

   $$p(t) = 1000 \times 2^{t/6}.$$

   How many hours does it take for the population to reach $250,000$?
3. (Stewart, Sec 1.6, Q.57) A bacteria population starts with 100 bacteria and doubles every three hours. So the number of bacteria after t hours can be modelled as $f(t) = 100 \times 2^{t/3}$. When will the population reach $50,000$?

Exercises.

1. Here is the graph of a quadratic polynomial. What is its equation?

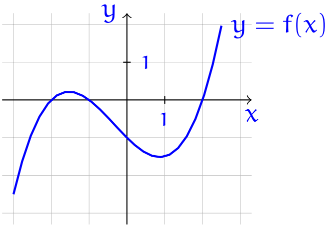


2. Find (a) $\log_{10} 1\,000\,000$, (b) $\log_2 1024$, (c) $\log_3 6561$.

3. The following graph is that of a cubic polynomial with equation

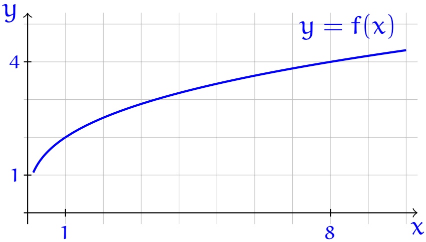
$$f(x) = K(x - a)(x - b)(x - c).$$

Find a, b, c, and K.



Exercises.

4. Here is the graph of the exponential function $f(x) = 2x^{1/n}$, where n is a natural number. What is the value of n?



5. (MA160/MA161 Paper 1, 2012/13) A biologist estimates that there are currently 500 zebra mussels in Lough Corrib, and that this number is doubling every year.
   (i) Show that the population can be modelled as $P(t) = 500 \times 2^t$, where t is time in years, and $t = 0$ represents the current time.
   (ii) Find P_0 and k such that the formula can be expressed as $P(t) = P_0 e^{kt}$.

6. Find, correct to 3 decimal places, (a) $\log_2 15$, (b) $\log_2 56.25$, (c) $\log_3 16$.