Week 3: Piecewise Defined Functions; Even and Odd Functions.

MA161/MA1161: Semester 1 Calculus.

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Tutorials.

Tutorials start from today (September 23).
You should attend **one of** the following:

1. Monday at 18:00 in IT204,
2. Tuesday at 11:00 in IT206,
3. Tuesday at 16:00 in ADB-1020,
4. Wednesday at 13:00 in CA101,
5. Wednesday at 18:00 in AM110,
6. Thursday at 13:00 in IT125G.

There is a 2nd sheet with problems to work on for download at

http://schmidt.nuigalway.ie/ma161-1

and at the end of these notes.

Recall . . .

Recall that a **function** is a rule that maps values from one set to another. In this course, we are mainly concerned with functions $f\colon D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$.

Given the formula for a function f, we frequently have to figure out:

- What is the **domain** of f? (The domain of f is the set of all numbers $x \in \mathbb{R}$ such that $f(x)$ makes sense.)
- What is the **range** of f? (The range of f is the set of all numbers $y \in \mathbb{R}$ such that $y = f(x)$ for some x in the domain of f.)

When figuring out the domain of f, we need to take into account:

- For which values x is $f(x)$ defined? In particular, we must **avoid** dividing by zero. The function $f(x) = \frac{x^2-9}{x-3}$ is not defined at $x = 3$.
- For which values x does f map x to a real number? For example, $\sqrt{x-3}$ is a real number only if $x - 3 \geqslant 0$.

Examples.

Example

What (subsets of $\mathbb{R}$) are the largest possible domain and range for the function $f(x) = \sqrt{x+2}$?

Example (MA160 (2014) Problem Sheet 1, Question 8)

What is the domain of the following real-valued function?

$$f(x) = \frac{17}{x^3 + 2x^2 - 8x}.$$

Example

Suppose that $f\colon D \to \mathbb{R}$ is a function given by $f(x) = 1 + x^3$.

1. If the domain D is the interval $[0, \infty)$, what is the range?
2. What domain gives the range $[0, \infty)$?

The Absolute Value Function.

Inequalities can be used inside the **definition** of a function.
This gives a **piecewise defined** function.
The most important example of a piecewise defined function is the **absolute value function**.

Definition (Absolute Value)

The **absolute value** of a real number x, denoted as $|x|$, is

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{else.} \end{cases}$$

Examples

- $|-17| = -(-17) = 17$.
- $|2019| = 2019$.
- $\{x \in \mathbb{R} : |x-3| < 1\} = (2,4)$.

Other Piecewise Defined Functions.

Example (The Heaviside Function)

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geqslant 0. \end{cases}$$

Example

Sketch the graph of the piecewise defined function

$$f(x) = \begin{cases} 1 + x, & \text{if } x < -1, \\ \frac{1}{2}x^2, & \text{if } x \geqslant -1. \end{cases}$$

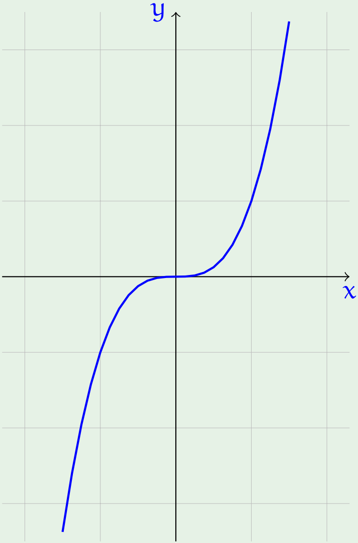
Example (MA161 Problem Set 1, Question 2)

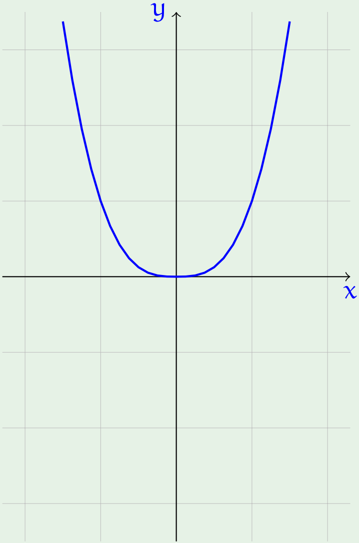
Solve the inequality

$$|x - 5| + 7 > 10.$$

Cubes.

Example ($f(x) = x^3$ vs. $g(x) = |x^3|$.)





Even and Odd Functions.

Definition

A function $f\colon D \to \mathbb{R}$ is called **even** if $f(-x) = f(x)$ for all x in its domain D.
A function $f\colon D \to \mathbb{R}$ is called **odd** if $f(-x) = -f(x)$ for all x in its domain D.

Most functions are **neither** even nor odd.

Example

- Show that the function $f(x) = x^2$ is even.
- Show that the function $f(x) = \dfrac{x^3 - x}{x^2 + 1}$ is odd.

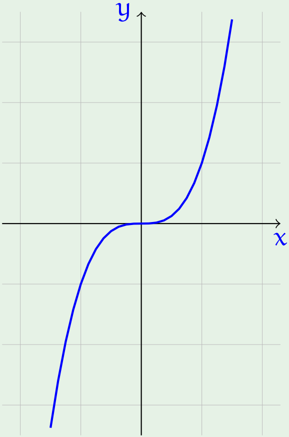
Example

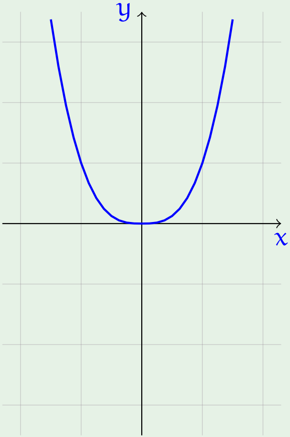
1. Is the function $f(x) = |x|$ even or odd?
2. Is the function $f(x) = |3 - x|$ odd or even?

Symmetries.

The notion of **even** and **odd** is related to **symmetries**.
Any even function is symmetric about the y-axis.
An odd function is symmetric about the origin $(0, 0)$.

Examples ($f(x) = x^3$ vs. $g(x) = |x^3|$.)





Even or Odd?

Example (MA160/MA161 Paper 1, 2012/13)

For each of the following functions, determine if it is even, odd or neither:

1. $$f(x) = \frac{1}{x} + x^7 + x^9;$$

2. $$g(x) = -5x^2 - 3x^4 - 2.$$

A Catalog of Functions.

see Section 1.2 of the Book . . .

We'll now spend some time reviewing the most common **essential functions**. These include:

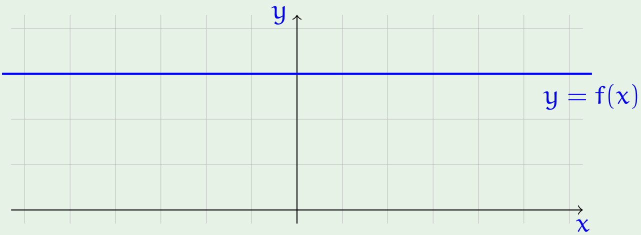
1. Linear Functions;
2. Polynomials;
3. Power Functions;
4. Rational Functions;
5. Algebraic Functions;
6. Trigonometric Functions;
7. Exponential Functions;
8. Logarithms.

0. Constant Functions.

But first, we look at the simplest possible functions.

Let $c \in \mathbb{R}$ be any real number. The **constant function** $f(x) = c$ assigns the same value c to all $x \in \mathbb{R}$.

Example (The graph of $f(x) = 3$)



- The constant function $f(x) = 0$ is the only function $f\colon \mathbb{R} \to \mathbb{R}$ which is even and odd at the same time . . .

1. Linear Functions.

A **linear function** is one whose graph is a straight line. It can be represented by a formula of the form

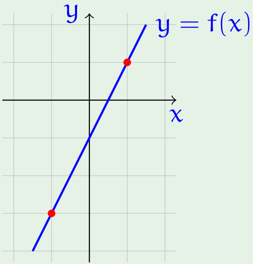
$$f(x) = mx + b,$$

where m is the **slope**, and b is the y**-intercept**.

Example

$f(x) = 2x - 1$.

Slope: 2.
y-intercept: -1.
$f(1) = 1$, $f(-1) = -3$, ...



Linear functions are easy to graph!

Exercises.

1. Find the largest possible domain and range for the following functions and sketch them:

   (i) $f(x) = x\,|x|$,

   (ii) $f(x) = \dfrac{1}{|x|+1}$,

   (iii) $f(x) = \dfrac{1}{|x+1|}$,

   (iv) $f(x) = \begin{cases} x+1, & \text{if } x < 0, \\ 1-x, & \text{if } x \geqslant 0, \end{cases}$

   (v) $f(x) = \begin{cases} x+9, & \text{if } x < -3, \\ -2x, & \text{if } |x| \leqslant 3, \\ -6, & \text{if } x > 3. \end{cases}$

2. Solve the following inequality: $|2x+5| + 20 \geqslant 25$.
3. For each of the following functions, determine if it is even, odd, or neither.

   (i) $f(x) = \dfrac{x}{x^2+1}$,

   (ii) $f(x) = \dfrac{x^2}{x^4+1}$,

   (iii) $f(x) = x|x|$,

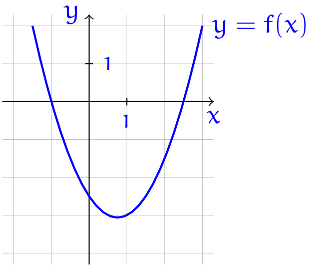
   (iv) $f(x) = 2 + x^2 + x^4$.

   (v) $f(x) = \dfrac{t^3+3t}{t^4-3t^2+4}$.

Exercises.

4. Are the trigonometric functions sin, cos, and tan even, odd, or neither?

5. Here is the graph of a quadratic polynomial. What is its equation?



6. The following graph is that of a cubic polynomial with equation

$$f(x) = K(x - a)(x - b)(x - c).$$

Find a, b, c, and K.