Week 3. Functions: Piecewise, Even and Odd.

MA161/MA1161: Semester 1 Calculus.

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Tutorials start from today (September 19).
You should attend one of the following:

1. Monday at 18:00 in IT202,
2. Tuesday at 11:00 in IT206,
3. Tuesday at 18:00 in AC201,
4. Wednesday at 14:00 in AC214,
5. Wednesday at 18:00 in AM104,
6. Thursday at 13:00 in IT125G.
MA161 Problem Set 1.

Deadline: 5pm, Friday, October 07, 2016.
Covers: Weeks 1-3, Calculus and Algebra.
Length: 15 Questions.
Attempts: 10.
Access: via blackboard.

See also: http://schmidt.nuigalway.ie/ma161
Credit: 5%.
Recall that a function is a rule that maps values from one set to another. In this course, we are mainly concerned with functions $f : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}$.

Given the formula for a function $f$, we frequently have to figure out:

- What is the **domain** of $f$? (The domain of $f$ is the set of all numbers $x \in \mathbb{R}$ such that $f(x)$ makes sense.)

- What is the **range** of $f$? (The range of $f$ is the set of all numbers $y \in \mathbb{R}$ such that $y = f(x)$ for some $x$ in the domain of $f$.)

When figuring out the domain of $f$, we need to take into account:

- For which values $x$ is $f(x)$ defined? In particular, we must avoid dividing by zero. The function $f(x) = \frac{x^2 - 9}{x-3}$ is not defined at $x = 3$.

- For which values $x$ does $f$ map $x$ to a real number? For example, $\sqrt{x - 3}$ is a real number only if $x - 3 \geq 0$. 
Examples.

Example
What (subsets of \( \mathbb{R} \)) are the largest possible domain and range for the function \( f(x) = \sqrt{x + 2} \)?

Example (MA160 (2014) Problem Sheet 1, Question 8)
What is the domain of the following real-valued function?

\[
  f(x) = \frac{17}{x^3 + 2x^2 - 8x}.
\]

Example
Suppose that \( f: D \to \mathbb{R} \) is a function given by \( f(x) = 1 + x^3 \).

1. If the domain \( D \) is the interval \([0, \infty)\), what is the range?
2. What domain gives the range \([0, \infty)\)?
The Absolute Value Function.

Inequalities can be used inside the definition of a function. This gives a piecewise defined function. The most important example of a piecewise defined function is the absolute value function.

Definition (Absolute Value)

The absolute value of a real number $x$, denoted as $|x|$, is

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{else}. \end{cases}$$

Examples

- $|-17| = -(-17) = 17$.
- $|2016| = 2016$.
- $\{x \in \mathbb{R} : |x - 3| < 1\} = (2, 4)$. 
Piecewise Defined Functions

Example (The Heaviside Function)

\[ H(t) = \begin{cases} 
0, & \text{if } t < 0, \\
1, & \text{if } t \geq 0.
\end{cases} \]

Example

Sketch the graph of the piecewise defined function

\[ f(x) = \begin{cases} 
1 + x, & \text{if } x < -1, \\
\frac{1}{2}x^2, & \text{if } x \geq -1.
\end{cases} \]

Example (MA161 Problem Set 1, Question 2)

Solve the inequality

\[ |x - 5| + 7 > 10. \]
Cubes.

Example \((f(x) = x^3 \text{ vs. } g(x) = |x^3|.)\)
Even and Odd Functions.

Definition

A function \( f: \mathbb{D} \rightarrow \mathbb{R} \) is called **even** if \( f(-x) = f(x) \) for all \( x \) in its domain \( \mathbb{D} \).

A function \( f: \mathbb{D} \rightarrow \mathbb{R} \) is called **odd** if \( f(-x) = -f(x) \) for all \( x \) in its domain \( \mathbb{D} \).

Most functions are **neither** even nor odd.

Example

- Show that the function \( f(x) = x^2 \) is even.
- Show that the function \( f(x) = \frac{x^3 - x}{x^2 + 1} \) is odd.

Example

1. Is the function \( f(x) = |x| \) even or odd?
2. Is the function \( f(x) = |3 - x| \) odd or even?
Symmetries.
The notion of **even** and **odd** is related to **symmetries**.
Any even function is symmetric about the $y$-axis.
An odd function is symmetric about the origin $(0, 0)$.

Examples ($f(x) = x^3$ vs. $g(x) = |x^3|$.)
Even or Odd?

Example (MA160/MA161 Paper 1, 2012/13)

For each of the following functions, determine if it is even, odd or neither:

1. \( f(x) = \frac{1}{x} + x^7 + x^9; \)

2. \( g(x) = -5x^2 - 3x^4 - 2. \)
A Catalog of Functions.

[Section 1.2 of the Book]

We’ll now spend some time reviewing the most common essential functions. These include:

1. Linear Functions;
2. Polynomials;
3. Power Functions;
4. Rational Functions;
5. Algebraic Functions;
6. Trigonometric Functions;
7. Exponential Functions;
8. Logarithms.
0. Constant Functions.

But first, we look at the simplest possible functions.

Let \( c \in \mathbb{R} \). The **constant function** \( f(x) = c \) assigns the same value \( c \) to all \( x \in \mathbb{R} \).

**Example (The graph of \( f(x) = 3 \))**

- The constant function \( f(x) = 0 \) is the only function \( f: \mathbb{R} \to \mathbb{R} \) which is even and odd at the same time . . .
1. Linear Functions.

A **linear function** is one whose graph is a straight line. It can be represented by a formula of the form

$$f(x) = mx + b,$$

where $m$ is the **slope**, and $b$ is the **$y$-intercept**.

**Example**

$$f(x) = 2x - 1.$$  
Slope: $2$.  
$y$-intercept: $-1$.  
$f(1) = 1$, $f(-1) = -3$, ...  

Linear functions are easy to graph!
Exercises.

1. Find the largest possible domain and range for the following functions and sketch them:

   (i) \( f(x) = x|x| \),

   (ii) \( f(x) = \frac{1}{|x| + 1} \),

   (iii) \( f(x) = \frac{1}{|x + 1|} \),

   (iv) \( f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ 1 - x, & \text{if } x \geq 0, \end{cases} \)

   (v) \( f(x) = \begin{cases} x + 9, & \text{if } x < -3, \\ -2x, & \text{if } |x| \leq 3, \\ -6, & \text{if } x > 3. \end{cases} \)

2. Solve the following inequality: \( |2x + 5| + 20 \geq 25 \).

3. For each of the following functions, determine if it is even, odd, or neither.

   (i) \( f(x) = \frac{x}{x^2 + 1} \),

   (ii) \( f(x) = \frac{x^2}{x^4 + 1} \),

   (iii) \( f(x) = x|x| \),

   (iv) \( f(x) = 2 + x^2 + x^4 \),

   (v) \( f(x) = \frac{t^3 + 3t}{t^4 - 3t^2 + 4} \).
Exercises.

4. Are the trigonometric functions \( \sin, \cos, \) and \( \tan \) even, odd, or neither?

5. Here is the graph of a quadratic polynomial. What is its equation?

6. The following graph is that of a cubic polynomial with equation
\[
f(x) = K(x - a)(x - b)(x - c).
\]
Find \( a, \ b, \ c, \) and \( K. \)