Week 2: Inequalities; Functions.

MA161/MA1161: Semester 1 Calculus.

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What, Where and When.

Lectures: Mon 1:00–1:50pm in the O'Flaherty (Calculus)
Tue 10:00–10:50am in the Kirwan (Calculus).

Tutorials: TBA

Web Site: There are online resources for MA161 Semester 1 Calculus at
`http://schmidt.nuigalway.ie/ma161-1`
and at this course's blackboard page.

Textbook: James Stewart
Calculus: Early Transcendentals 7th Edition.

Solving Inequalities.

We'll solve some inequalities, starting with simple **linear** examples.

1. Find all numbers x such that $5x + 1 > 0$.
   (Soln: $x \in (-\frac{1}{5}, \infty)$.)

2. Find all numbers x such that $-\frac{1}{2}x + 4 \leqslant 0$.
   (Soln: $x \in [8, \infty)$.)

3. Find all numbers x such that $3x + 1 \geqslant 4x + 2$.
   (Soln: $x \in (-\infty, -1]$.)

More Inequalities.

Next, we'll solve some more complicated **nonlinear** inequalities. Here it is helpful to remember that, for two numbers a and b,

- if $ab = 0$ then at least one of a and b equals 0;
- if $ab < 0$ then one of a and b is positive and one is negative;
- if $ab > 0$ then either both numbers are positive, or both are negative.

1. Find all real numbers x such that $x^2 \geqslant 0$.
2. Find all real numbers x such that $x^3 \leqslant 0$.
3. Find all real numbers x such that $x^3 + x^2 < 0$.
   (Soln: $\{x : x < -1\} = (-\infty, -1)$.)
4. Find all real numbers x such that $x^2 - 1 \geqslant 0$.
5. Find all real numbers x such that $x^2 + 1 \leqslant 0$.

Example.

Example (MA161 Semester 1 Exam, 2013/2014.)

Solve the following inequality: $x^2 > x + 6$.

Solution:

1. Write the inequality as $x^2 - x - 6 > 0$.
2. Solve the quadratic equation: $x^2 - x - 6 = 0$ iff $x = 3$ or $x = -2$.
3. Factorize the quadratic: $x^2 - x - 6 = (x - 3)(x + 2)$.
4. if $ab > 0$ then EITHER $a > 0$ and $b > 0$, OR $a < 0$ and $b < 0$: EITHER $x - 3 > 0$ and $x + 2 > 0$, OR $x - 3 < 0$ and $x + 2 < 0$.
5. EITHER $x > 3$ and $x > -2$, OR $x < 3$ and $x < -2$.
6. Either $x > 3$ or $x < -2$.
7. Solution: $x \in (-\infty, -2) \cup (3, \infty)$.

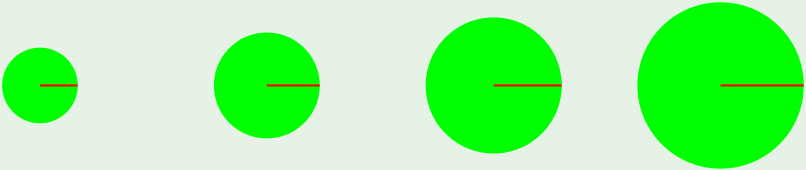
Functions: Domain, Codomain, Range.

see Section 1.1 of the Book . . .

The idea of a **function** is one of the **key concepts** in mathematics.

Example

The **area** A of a circle **depends** on the **radius** r of the circle.
The **rule** that connects r and A is given by the **equation** $A = \pi r^2$.
This **formula** assigns to each positive number r one value of A.
We say that A is a **function** of r, and write $A(r) = \pi r^2$.



Functions: Domain, Codomain, Range.

Definition (Function)

A **function** f is a rule that assigns to **each** element x in a set X exactly one element, called $f(x)$, in a set Y. We write

$$f\colon X \to Y$$

and say that "f is a function from X to Y". Here, X is called the **domain** of the function f, and Y is called the **codomain** of f. The **range** of f is $\{f(x) : x \in X\}$, the set of all possible values $f(x)$ as x varies over the domain.

- The number $f(x)$ is called the **value** of f at x (or the **image** of x under f) and pronounced as "f of x".
- Not every element $y \in Y$ needs to occur as value of some $x \in X$, only the elements in the range of f do.
- One element $y \in Y$ can serve as value $f(x)$ for several elements $x \in X$.

Examples of Functions.

Examples

1. The **identity** function f, defined by the rule
$$f(x) = x,$$
has $\mathbb{R}$ as its domain and codomain. Its range is also $\mathbb{R}$.
More precisely, every $x \in \mathbb{R}$ occurs exactly once as a value for f (of x itself).
2. The **square** function
$$f(x) = x^2$$
has $\mathbb{R}$ as its domain and codomain. Its **range**, however is $[0, \infty)$, as $x^2 \geqslant 0$ for all $x \in \mathbb{R}$. More precisely, 0 is the square of $x = 0$ (only), and every number $y > 0$ arises as the square of two different numbers x and $-x$.
3. The **square root** function $f(x) = \sqrt{x}$ has $[0, \infty)$ as its domain and range.

4 Ways to Represent a Function.

A function can be represented in many differents ways:

1. **verbally** (by a description in **words**);
2. **numerically** (as a table of **values**);
3. **visually** (as a **graph**);
4. **algebraically** (by an explicit **formula**).

Example (Verbally)

Consider the function a that

“assigns to each **person** in this room their **age** (in years)”.

Its **domain** is “students in MA161/MA1611 and their lecturer”.
Its **codomain** is $\{0, 1, 2, 3, \ldots, 120\} \subseteq \mathbb{N}$, the **possible** ages of people.
Its **range** is something like $\{17, 18, 19, 21, 32, 43, 54\}$, the set of the **actual** ages of the people in the room.

Tabular Representation of Functions.

Example (Table)

The function g assigns to each **percentage** value an exam **grade**, according to the following table.

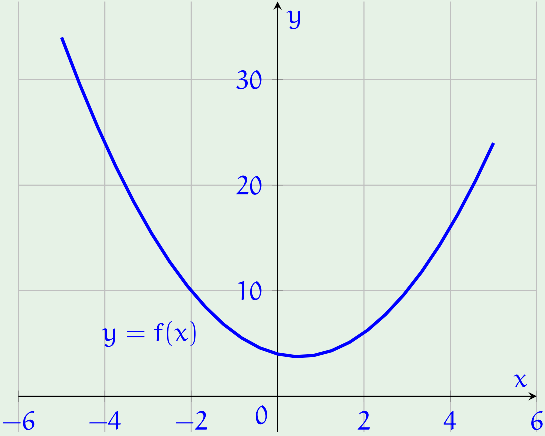
Percentage	Grade
70 – 100	A
60 – 69	B
55 – 59	C+
50 – 54	C−
40 – 49	D
35 – 39	E+
30 – 34	E−
0 – 29	F

- The **domain** is the set of percentages $\{0, 1, 2, \ldots, 100\}$.
- The **codomain** and **range** are the set of grades $\{\mathtt{A}, \mathtt{B}, \mathtt{C+}, \mathtt{C-}, \ldots, \mathtt{F}\}$

Graphical Representation of Functions.

Functions can be **visualized** in a variety of ways. The most common way to visualize a function $f\colon D \to \mathbb{R}$ is its **graph** $\{(x, y) : x \in D,\ y = f(x)\} \subseteq \mathbb{R}^2$ in the x, y-plane.

Example (Function Graph)



Algebraic Representation of functions.

Example (Formula)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by the **formula**

$$f(x) = 2x^3 - 1.$$

Then we can **compute** some values:

$$\begin{aligned} f(0) &= -1, & f(1) &= 1, & f(10) &= 1999, \\ f(-1) &= -3, & f(-10) &= -2001, & &\dots \end{aligned}$$

The **domain** and **codomain** of f are both $\mathbb{R}$, all the real numbers (by definition).
And it looks like the **range** also is all of $\mathbb{R}$, or is it?

- The formula of a function f can be used to determine (parts of) its graph . . .

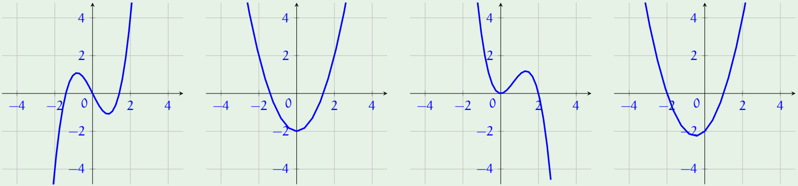
Telling a Function from its Graph.

Example (MA161 Paper 1, 2012/2013)

The four graphs below represent the following polynomials:

(i) $x^2 - 2$, (ii) $x^2 + x - 2$, (iii) $x^3 - 2x$, and (iv) $-x^3 + 2x^2$,

not necessarily in that order. Determine which polynomial corresponds to which graph, and give an explanation to support your answer.



Deciding on the Domain and Range.

Technically, the domain and codomain form part of a function definition, i.e., in order to specify a function we should also specify its domain and codomain explicitly.
In practice, however, many functions are simply given by a formula, and their domain is understood to be as large as possible.
Determining the (largest possible) domain and range from a formula is therefore an important skill . . .

Examples

- Find the largest possible subsets of $\mathbb{R}$ that are the domain D and the range for the function $f\colon D \to \mathbb{R}$ given by the formula
$$f(x) = x^2 + 1.$$
- What (subsets of $\mathbb{R}$) are the largest possible domain and range for the function $f(x) = \sqrt{x+2}$?
- Determine the (largest possible) domain and range for the function
$$f(x) = (x^2 - x)^{-1}.$$

Domain and Range.

Example

Suppose that a function $f\colon D \to \mathbb{R}$ is given by the formula

$$f(x) = 1 + x^2.$$

1. If the domain D is the set $[0, \infty)$, what is the range?
2. What choice of domain gives $[0, \infty)$ as the range?

In many cases the largest possible domain of a function $f\colon D \to \mathbb{R}$ depends on

- for what values of x is $f(x)$ defined? For example, the function $f(x) = 1/x$ is not defined at $x = 0$.
- for what values of x does $f(x)$ evaluate to a real number? For example, since $\sqrt{x}$ is a real number only if $x \geqslant 0$, the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$.

Absolute Value Functions.

Inequalities can also be used inside the **definition** of a function.
This gives a **piecewise defined** function.
The most important example of a piecewise defined function is the **absolute value function**.

Definition (Absolute Value)

The **absolute value** of a real number x, denoted as $|x|$, is

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{else.} \end{cases}$$

Examples

Sketch the graphs of the following functions:

1. $f(x) = |x|$; 2. $f(x) = |x - 1|$; 3. $f(x) = |x| - 1$.

Exercises.

1. For each of the following functions, determine the largest possible domain and range as subsets of $\mathbb{R}$.
   (i) $f(t) = 1/(1+t)$,
   (ii) $f(x) = \sqrt{9 - x^2}$,
   (iii) $f(x) = \cos(x)$,
   (iv) $f(t) = \sin(5t - 2)$,
   (v) $f(x) = 1 + (1 - x^2)^{-1}$,
   (vi) $f(x) = e^x$.
2. Find the domain of the following functions:
   (i) $f(x) = \dfrac{x}{3x - 1}$,
   (ii) $f(x) = \dfrac{1 + x}{x^2 - 3x + 2}$,
   (iii) $f(x) = \sqrt{4 - x^2}$,
   (iv) $f(x) = \sqrt{x} + \sqrt{4 - x}$.
3. Find the largest possible domain and range for the following functions and sketch them:
   (i) $f(x) = x\,|x|$,
   (ii) $f(x) = \dfrac{1}{|x| + 1}$,
   (iii) $f(x) = \dfrac{1}{|x + 1|}$,
   (iv) $f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ 1 - x, & \text{if } x \geqslant 0, \end{cases}$
   (v) $f(x) = \begin{cases} x + 9, & \text{if } x < -3, \\ -2x, & \text{if } |x| \leqslant 3, \\ -6, & \text{if } x > 3. \end{cases}$

Exercises.

4. (MA101 Summer Exam 2011/2012) The four graphs below are of the following functions: $y = 2^x$, $y = x^{-2}$, $y = \sqrt{x}$ and $y = x^2$, but not necessarily in that order. Which is which?