Week 12: Exponential Growth and Decay

MA161/MA1161: Semester 1 Calculus.

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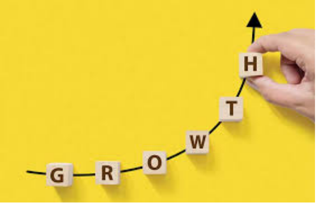


Exponential Growth and Decay.

- Many quantities in Nature grow or decay at **a rate proportional to their size**.

- If $P(t)$ is the number of individuals in a **population**, then one can expect the rate of growth $P'(t)$ to be proportional to the **size** $P(t)$.
- In **Nuclear Physics**, the mass of a **radioactive substance** decays at a rate proportional to the **mass**.
- In **Chemistry**, the rate of a **unimolecular first-order reaction** is proportional to the **concentration** of the substance.
- In **Finance**, the value of a **savings account** with continuously compounded interest increases at a rate proportional to the **current value**.
- In a **Pandemic**, a **virus** at first will spread at a rate proportional to the number of **infected people** . . .

The Law of Natural Growth/Decay.

- In general, if $y = f(t)$ is the value of a quantity y at time t, and if the rate of change of y wrt. t is proportional to y then
$$y' = ky, \quad \text{for some constant } k.$$
- This type of equation is called a **differential equation**.
- We **know** a solution: $f(t) = Ce^{kt}$ has $f'(t) = Cke^{kt} = kf(t)$.
- It's the **only solution**: Recall $(\ln y)' = \frac{y'}{y}$. Take **antiderivatives**:
$$\frac{y'}{y} = k \text{ gives } \ln y = kt + D, \text{ and } y = Ce^{kt},$$
where $e^D = C = Ce^{k \cdot 0} = y(0)$, the **initial value**.

- **Theorem.** The only solution of the differential equation $y' = ky$ with initial value $y(0) = C$ is the exponential function
$$y = Ce^{kt}.$$

Population Growth.

- Denote by $P(t)$ the size of a population at time t.
- Assuming that the population grows at a rate proportional to its size, we have

$$P' = kP, \text{ or } \frac{P'}{P} = k.$$

- The quantity $k = P'/P$ is called the **relative growth rate**.

- The **relative growth rate** of a population that grows at a rate proportional to its size is **constant**.

- The relative growth rate k forms part of the solution $P(t) = Ce^{kt}$.

- **Example.** Suppose a population P at time 0 has size P_0 and grows at a relative rate of 2%. Then $P(t) = P_0\, e^{0.02t}$.

World Population.

- **Example.** The world population in 1980 was 4458 million, and in 2020 it is 7831 million. Assuming this grows at a rate proportional to its size, determine the relative growth rate. Formulate a model for the world population and use it to estimate the world population in 2000, and to predict the population in the year 2050.
- **Solution:** Measure time t in years and let $t = 0$ in 1980. Then $P(0) = 4458$ and $P(40) = 7831$. By the **Theorem** $P(t) = P(0)e^{kt}$:

$$e^{40k} = \frac{P(40)}{P(0)} = \frac{7831}{4458} \implies k = \frac{1}{40} \ln \frac{7831}{4458} \approx 0.014085$$

- With a **relative growth rate** of about 1.4%, the **model** is

$$P(t) = 4458\, e^{0.014085t} = 4458 \cdot (1.014185)^t.$$

- In 2000, it gives $P(20) = 4458 \cdot (1.014185)^{20} \approx 5909$ million.
  (The **actual** figure was 6143 million.)
- In 2050, it gives $P(70) = 4458 \cdot (1.014185)^{70} \approx 11949$ million.
  (About **double** the size of 50 years before . . .)

Newton's Law of Cooling.

- A warm body **cools down** at a rate proportional to the **difference in temperatures** between the body and its surroundings.

- Denote by $T(t)$ the temperature at time t, and by T_s the temperature of the surroundings. Then
$$T' = k(T - T_s) \text{ for some constant } k.$$
- Setting $y = T - T_s$, we have $y' = T'$ and we can solve $y' = ky$ for $y = y_0 e^{kt}$. Thus $T - T_s = (T_0 - T_s)e^{kt}$.

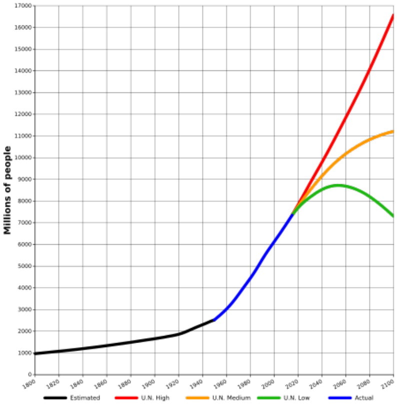
- **Example.** A glass of milk at $22°$C is placed into a fridge at $7°$C. After 30 minutes the milk has cooled to $16°$C.

  How long does it take for the milk to cool down to $10°$C?
- **Solution:** $T_0 - T_s = 15$ and $T(30) - T_s = 9 \implies e^{30k} = \frac{9}{15} = 0.6$.
- Hence $k = \frac{1}{30} \ln 0.6 \approx -0.01703$.
- $15e^{-0.01703t} = 10 - 7 \implies -0.01703t = \ln 0.2 \implies t \approx 94.5$.

The **Truth** about Exponential Growth.

- 12 billion people by 2050. Really?
- In practice, exponential growth (forever) doesn't exist.
- Eventually something will stop your growth (finite resources).
- Rather than $y' = ky$, we have $y' = ky(1-y)$:

  Quantity y grows at a rate that is proportional (k) to its current size (y) **and to the currently remaining resources** ($1-y$).



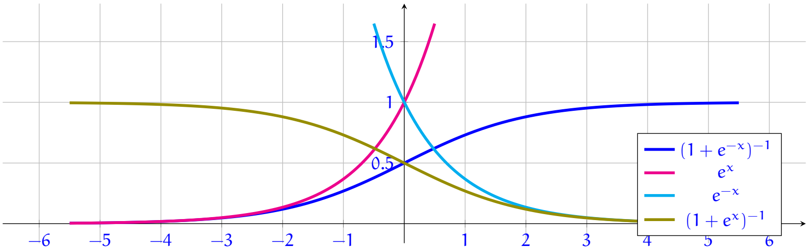
Logistic Equation

- For $k = 1$ and $y_0 = \frac{1}{2}$, the **logistic equation**

$$y' = y(1 - y)$$

has an exact solution

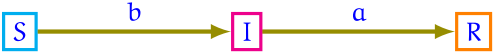
$$y = \frac{e^x}{e^x + 1} = \frac{1}{1 + e^{-x}} \text{ (Check!)}$$



- More generally, $y' = ry(1 - y/K)$, with a **growth rate** r and **carrying capacity** K yields $y = K(1 + (\frac{K - y_0}{y_0})e^{-rx})^{-1}$.

Modelling a Pandemic.

- People are either **S**usceptible, **I**nfected or **R**ecovered.



- Here, $S + I + R = N$ is **constant**, b is the **contact rate**, and a is the **cure rate** corresponding to an **infectious period** of $1/a$.

- **SIR Model**:

$$S' = -\frac{b}{N} SI, \qquad I' = \frac{b}{N} SI - aI, \qquad R' = aI$$

- This set of differential equations does not have an exact solution.
- Still: When is $I \searrow$? **Recall:** $I \searrow \iff I' < 0$. From **SIR**:

$$I' = \frac{b}{N} SI - aI < 0 \iff R_0 := b/a < N/S.$$

- $\rightsquigarrow 3$ ways to decrease R_0: $b \searrow$, $a \nearrow$, $S \searrow$.

Discrete Pandemic.

- What is $\lim_{t\to\infty} S(t)$?
- Replace S' by a difference quotient $\frac{S_{t+1}-S_t}{\Delta t} = S_{t+1} - S_t$, $\Delta t = 1$.
- The **differential equations** then become **difference equations**:

$$S_{t+1} - S_t = -\frac{b}{N} S_t I_t, \quad I_{t+1} - I_t = \frac{b}{N} S_t I_t - a I_t, \quad R_{t+1} - R_t = a I_t$$

- Here, $-(S_{t+1} - S_t) = \frac{b}{N} S_t I_t$ is the daily number of **new cases**.
- And $N - S_t$ is the **total number of cases** at time t.
- We can **simulate** the pandemic **numerically**.

- **Example.** Let $N = 1000$, $I_0 = 3$, $S_0 = 997$, $R_0 = 0$.
- Suppose that $a = 0.04$, $b = 0.09$. Then:

  $S_1 = S_0 - \frac{b}{N} S_0 I_0 = 997 - 0.00009 \cdot 997 \cdot 3 \approx 996.73$,

  $I_1 = I_0 + \frac{b}{N} S_0 I_0 - a I_0 = 3 + 0.00009 \cdot 997 \cdot 3 - 0.04 \cdot 3 \approx 3.15$,

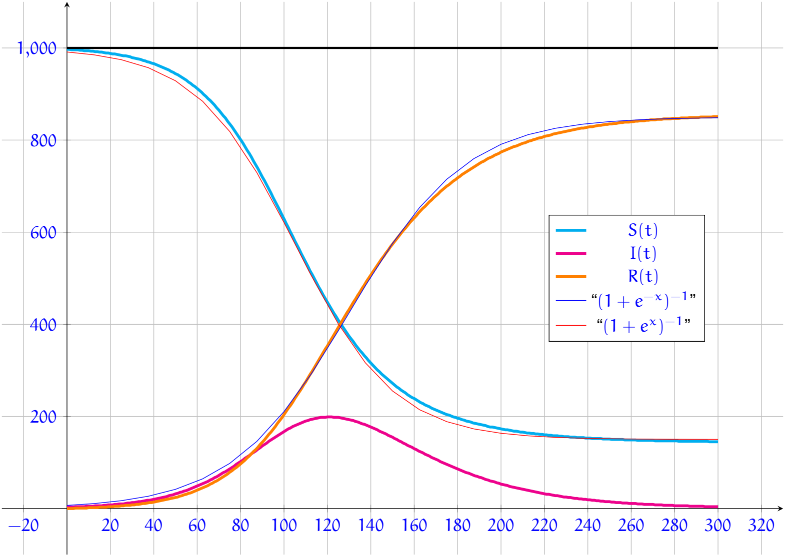
  $R_1 = R_0 + a I_0 = 0 + 0.04 \cdot 3 = 0.12$.
- Wash, rinse, repeat . . .

A Python Program and its Output.

```
##  A simple SIR model

# parameters
a, b = 0.04, 0.09
sir = (0, 997, 3, 0)
N = sum(sir)

# model
def next(sir):
    n, s, i, r = sir
    ss = s - b/N*s*i
    ii = i + b/N*s*i - a*i
    rr = r + a*i
    return (n+1, ss, ii, rr)

# generate csv
print("n,s,i,r")
print("%.f,%.f,%.f,%.f" % (sir))
for i in range(300):
    sir = next(sir)
    print("%.f,%.f,%.f,%.f" % (sir))
```

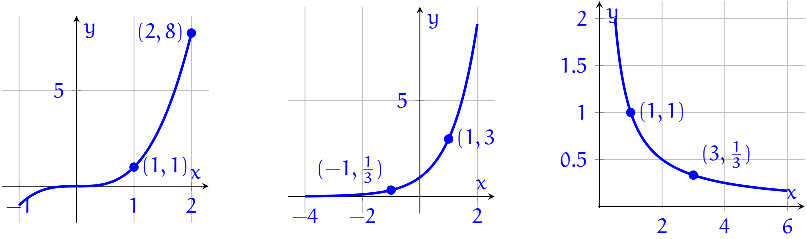
```
n,s,i,r
0,997,3,0
1,997,3,0
2,996,3,0
3,996,3,0
4,996,4,1
5,996,4,1
6,995,4,1
7,995,4,1
8,994,4,1
9,994,5,1
10,994,5,2
11,993,5,2
12,993,5,2
13,992,6,2
14,992,6,2
15,991,6,3
16,991,6,3
17,990,7,3
18,989,7,3
19,989,8,4
20,988,8,4
...
```

Graph


1,000
800
600
400
200
−20
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
S(t)
I(t)
R(t)
“(1 + e−x)−1”
“(1 + ex)−1”

Revision Exercises Q1.

(a) The three graphs below are of the functions $P(x) = a^x$, $Q(x) = x^b$ and $R(x) = x^{-c}$, where a, b and c are positive integers. Determine which graph corresponds to which function, and find a, b and c.



(b) Solve the following inequalities:

(i) $18 - 2x^2 \leqslant 0$. (ii) $1 + 2|x - 3| > 7$.

Revision Exercises Q1.

(c) According to Wikipedia, the isotope radon-222 has a half-life of 4 days, which means that, compared to time t (days), only half of the substance remains at time $t + 4$.

(i) If there are 100g present at time $t = 0$, show that the quantity remaining at time t can be described by the function

$$f(t) = 100 \cdot (\tfrac{1}{2})^{t/4}.$$

(ii) How many days must pass for the mass to reduce from 100g to less than 1g?

(d) For each of the following functions, determine if it is even, odd, or neither.

(i) $f(x) = -1 - x^2 - x^4$.

(ii) $f(x) = e^x - e^{-x}$.

(e) Evaluate the following limits:

(i) $\displaystyle\lim_{x\to-8} \frac{x^2 + 11x + 24}{x + 8}$;

(ii) $\displaystyle\lim_{x\to\infty} \frac{x^4 - 4x^2 + 4}{x^5 + 4x^3}$.

Revision Exercises Q2.

(a) (i) Determine if the function $f(x) = \dfrac{x^3}{|x|}$ is continuous at $x = 0$.

(ii) Determine if the function

$$f(x) = \begin{cases} 4x^2 - 2x + 1, & x \leqslant -1, \\ 6 - x, & -1 < x \leqslant 1, \\ 1 + (1 + x)^2, & x > 1, \end{cases}$$

is continuous for all $x \in \mathbb{R}$.

(b) Use that $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to show that the derivative of $f(x) = 2\sqrt{x}$ is $f'(x) = \dfrac{1}{\sqrt{x}}$.

(c) Find the derivatives of the following functions:

(i) $f(x) = \ln(x^2 + 1)$;

(ii) $f(x) = \dfrac{\cos(x^3 + 2)}{x + 1}$.

Revision Exercises Q2.

(d) Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

(e) A chemist models the temperature of a reacting mixture in an experiment as

$$f(t) = 10 + 5t - \ln(1 + 40t),$$

where t is time in minutes, and temperature is measured in degrees Celsius. The experiment runs for one minute: from $t = 0$ to $t = 1$.

(i) Find the maximum and minimum temperature of the mixture during the experiment.

(ii) Give a sketch of the graph of f, showing clearly the regions where the temperature is increasing, and where it is decreasing.