Week 11: Optimization

MA161/MA1161: Semester 1 Calculus.

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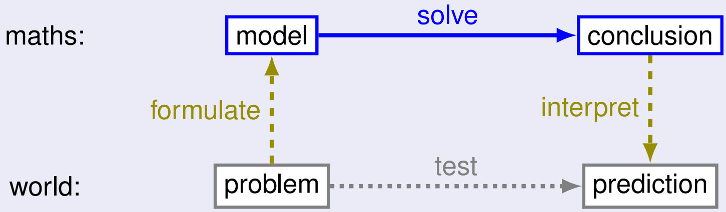
Optimization Problems.

- Methods for **finding extreme values** have many **practical applications**.

- Businesspeople **minimize costs** and **maximize profits**.
- In Optics, **Fermat's Principle** states that **light** follows the path that takes the **least time**.

- **Recall:** How a **mathematical model** can **solve a problem**.



Solving Optimization Problems.

1. **Understand the Problem.** **Read** the problem **carefully**. What is the **unknown**? What are the **given quantities**? What are the given **conditions**?
2. **Draw a Diagram.** For most optimization problems it is helpful to draw a diagram, and to **identify** the given and the required **quantities** on the diagram.
3. **Introduce Notation.** Assign a **symbol** (e.g., Q) to the quantity to be **optimized**. Select symbols (a, b, c, . . .) for **other unknown quantities**. Often **initials** are good suggestive symbols:

   A for area, h for height, t for time.

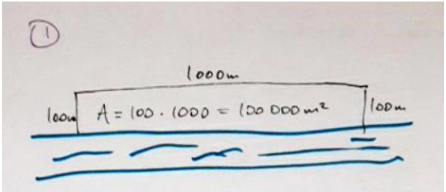
   Label the diagram with these symbols.
4. Express Q in terms of other symbols.
5. If Q now is a function of more than one variable, use the given information to **eliminate variables**.
6. Use **Calculus** to find the absolute maximum or minimum.

Fence.

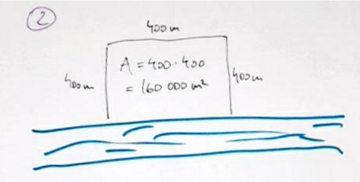
1. Read and understand:

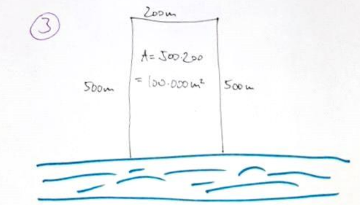
- **Example.** A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river.
- What are the dimensions of the field that has the largest area?

2. Draw a diagram . . .



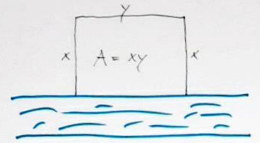
. . . or two or three:





Solution.

3. Introduce notation:



- We wish to **maximize** the **area** A of the rectangle.

4. Express A in terms of the other symbols: $A = xy$.
5. Eliminate the variable y. Total length of the fence:

$$2x + y = 1200.$$

- Hence $y = 1200 - 2x$ and so

$$A = x(1200 - 2x) = 1200x - 2x^2.$$

- $x, y \geqslant 0 \implies 0 \leqslant x \leqslant 600$.

6. Maximize $A(x) = 1200x - 2x^2$ for $x \in [0, 600]$.

- **Closed Interval Method**:
- Critical Points: $A'(x) = 0$.

$A'(x) = 1200 - 4x$.

$1200 - 4x = 0 \Leftrightarrow x = 300$

$A(300) = \underline{180\,000}$

- End Points: $A(0), A(600) = \underline{0}$.
- **A:** The largest field has area

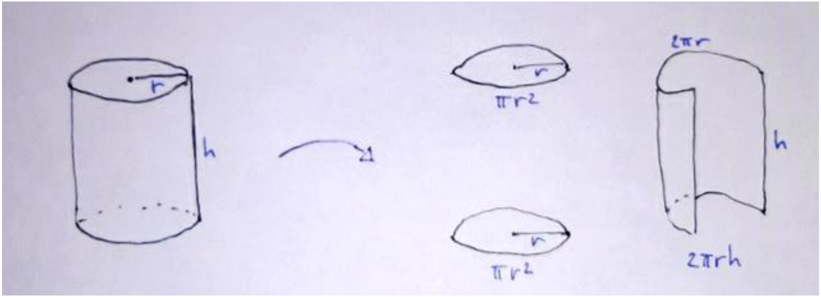
$$\underline{A(300) = 180\,000}.$$

- Alternatively, $A''(x) = -4 < 0$ implies that $A(x)$ is always **concave down** ($\frown$), and so the local maximum at $x = 300$ (and $y = 1200 - 2 \cdot 300 = 600$) must be a global one.

Can.

- **Example.** A cylindrical can is to be made to hold 1 l of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

- **Solution:** Start with a diagram, introducing notation.



Solution.

- In order to minimize the cost, we minimize the **total surface area**
$$A = 2\pi r^2 + 2\pi r h.$$
- The volume $\pi r^2 h$ of the can is 1000 cm^3; **eliminate** $h = \frac{1000}{\pi r^2}$:
$$A(r) = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$
- Since $r > 0$, we want to **minimize** the function
$$A(r) = 2\pi r^2 + \frac{2000}{r} = \frac{2\pi r^3 + 2000}{r}, \quad r \in (0, \infty).$$
- (Check some values: $A(1) = 2006.28$ and $A(10) = 828.32$.)
- Differentiate to find the **critical points**: $A'(r) = \frac{4\pi r^3 - 2000}{r^2}$.
- Solve for r: $A'(r) = 0 \Leftrightarrow \pi r^3 = 500 \Leftrightarrow r = r_0 = 5\sqrt[3]{4/\pi} \approx \underline{5.4193}$.
- $A'(r) < 0$ and $A(r) \searrow$ for $r < r_0$; $A'(r) > 0$ and $A(r) \nearrow$ for $r > r_0$: Hence the **local minimum**
$$A(r_0) = \frac{3000}{5\sqrt[3]{4/\pi}} = 300\sqrt[3]{2\pi} \approx \underline{553.58}$$
is a **global minimum**.
- Also, $h_0 = \frac{1000}{\pi r_0^2} = \frac{1000}{\pi(5\sqrt[3]{4/\pi})^2} = 10\sqrt[3]{4/\pi} = 2r_0$.

Afterthoughts.

- **First Derivative Test for Absolute Extreme Values.** If c is a critical point of a continuous function defined on an interval and:

(a) if $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$ then $f(c)$ is the **absolute maximum** value of f.

(b) if $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$ then $f(c)$ is the **absolute minimum** value of f.

- **Alternatively**, **Implicit Differentiation** (wrt. r) of

$$A = 2\pi r^2 + 2\pi rh, \quad \pi r^2 h = 1000$$

yields (**Product Rule**)

$$A' = 4\pi r + 2\pi rh' + 2\pi h, \quad \pi r^2 h' + 2\pi rh = 0.$$

- At the minimum $A' = 0$, so

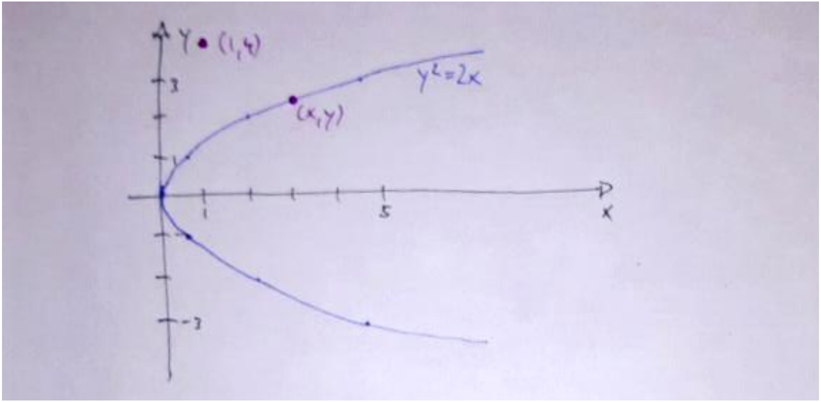
$$2r + rh' + h = 0, \quad rh' + 2h = 0$$

and $h = 2r$. (Then $\pi r^2 h = 1000$ yields $r = 5\sqrt[3]{4/\pi}$.)

Parabola.

- **Example.** Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

- **Solution:** Start with a diagram . . .



Solution.

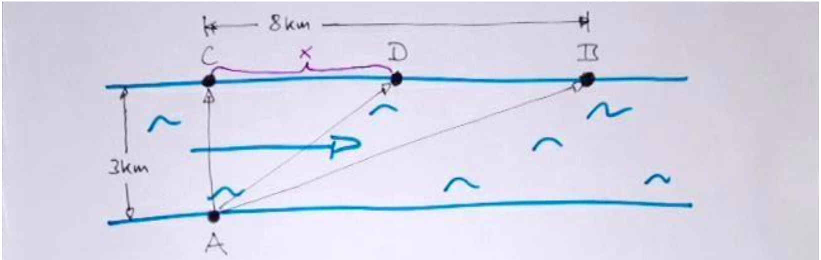
- The distance between $(1, 4)$ and (x, y) is
$$d = \sqrt{(x-1)^2 + (y-4)^2}.$$
- But if (x, y) lies on the parabola then $x = \frac{1}{2}y^2$ and
$$d = \sqrt{(\tfrac{1}{2}y^2 - 1)^2 + (y-4)^2}.$$
- Minimize
$$f(y) = d^2 = (\tfrac{1}{2}y^2 - 1)^2 + (y-4)^2, \quad y \in \mathbb{R}.$$
- Differentiation (wrt. y) yields (**Chain Rule**):
$$f'(y) = 2(\tfrac{1}{2}y^2 - 1)y + 2(y-4) = y^3 - 8.$$
- So $f'(y) = 0$ for $y_0 = 2$ and (**First Derivative Test for Absolute Extreme Values**) $f(2) = 5$ is the **absolute minimum value**.
- The corresponding x-coordinate is $x_0 = \frac{1}{2}y_0^2 = 2$.
- **A:** $(2, 2)$ is closest to $(1, 4)$ with distance $d_0 = \sqrt{f(2)} = \sqrt{5}$.

River.

- **Example.** A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank as quickly as possible.

  He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B.

  If he can row 6 km/h and run 8 km/h where should he land to reach B as soon as possible?



Solution.

- Let x be the distance from C to D. Running distance: $8 - x$; Rowing distance: $\sqrt{x^2 + 3^2}$ (**Pythagoras**).
- Using "**speed = distance/time**", running time is $(8 - x)/8$ and rowing time is $\sqrt{x^2 + 9}/6$.
- Minimize total time

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}, \quad x \in [0, 8].$$

- Differentiate:

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

- Critical Point ($x \geqslant 0$):

$$T'(x) = 0 \Leftrightarrow 4x = 3\sqrt{x^2 + 9} \Leftrightarrow 7x^2 = 81 \Leftrightarrow x = 9/\sqrt{7}.$$

- **Closed Interval Method**:

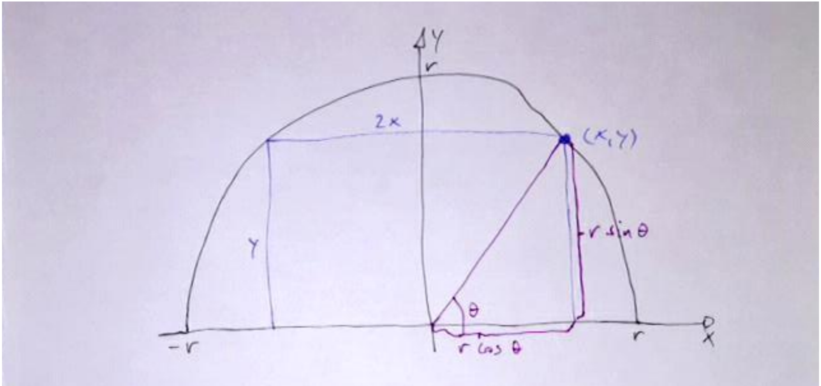
$T(0) = 1.5, \quad T(9/\sqrt{7}) = 1 + \sqrt{7}/8 \approx 1.33, \quad T(8) = \sqrt{73}/6 \approx 1.42$

- **A:** Absolute minimum occurs at $x = 9/\sqrt{7} \approx 3.4$ km downstream.

Semicircle.

- **Example.** Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

- Diagram:



Solution.

- The semicircle is the upper half of the circle $x^2 + y^2 = r^2$.
- The rectangle has sides of length $2x$ and y, and area $A = 2xy$.
- Eliminate $y = \sqrt{r^2 - x^2}$: Maximize

$$A(x) = 2x\sqrt{r^2 - x^2}, \quad x \in [0, r].$$

- Differentiate (**Product Rule**, **Chain Rule**):

$$A'(x) = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}.$$

- $A'(x) = 0 \Leftrightarrow x = r/\sqrt{2}$ gives maximum value ($A(0) = A(r) = 0$).
- **A:** The **largest inscribed rectangle** has area

$$A(r/\sqrt{2}) = 2(r/\sqrt{2})\sqrt{r^2 - (r/\sqrt{2})^2} = \underline{r^2}.$$

- **Solution 2:** Express the area A in terms of the angle θ:

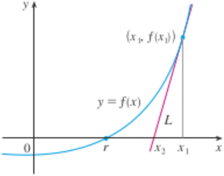
$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\,\cos\theta) = r^2\sin(2\theta).$$

- We know that $\sin(2\theta)$ has maximum value 1 at $2\theta = \pi/2$.
- $A(\theta)$ has maximum value r^2 at $\theta = \pi/4$. (No Calculus needed!)

Newton's Method.

- How to solve an equation like $48x(1+x)^{60} - (1+x)^{60} + 1 = 0$?

- To solve $f(x) = 0$, find an **x-intercept** r of the graph of f.
- Start with a **first approximation** x_1 for r.
- The equation of the tangent to the curve $y = f(x)$ at $(x_1, f(x_1))$ is
$$y - f(x_1) = f'(x_1)(x - x_1)$$
- The tangent intercepts the x-axis at $(x_2, 0)$, so
$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$
- Solving for x_2 yields
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$
a second, **better approximation** for r.
- This process can be repeated.



- **Newton's Method:** $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad r = \lim_{n\to\infty} x_n.$

Solving a Cubic.

- **Example.** Starting with $x_1 = 2$, find the 3rd approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

- **Solution:** Set $f(x) = x^3 - 2x - 5$. Then $f'(x) = 3x^2 - 2$.
- **Newton's Method**: $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$.
- $x_1 = 2$ gives $x_2 = x_1 - \dfrac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} = 2 - \dfrac{2^3 - 2 \cdot 2 - 5}{3 \cdot 2^2 - 2} = 2.1$
- And $x_3 = x_2 - \dfrac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} = (2.1) - \dfrac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2}$

  ≈ 2.0946, accurate to 4 decimal places.

Extracting 6th Roots.

- If successive approximations x_n and x_{n+1} **agree** to 8 decimal places, then x_{n+1} usually **is** accurate to 8 decimal places.

- **Example.** Find $\sqrt[6]{2}$ correct to eight decimal places.
- **Solution:** $\sqrt[6]{2}$ is a root of $f(x) = x^6 - 2$.
- Differentiate: $f'(x) = 6x^5$.
- **Newton's Method**: $x_{n+1} = x_n - \dfrac{x_n^6 - 2}{6x_n^5} = \frac{5}{6}x_n + \frac{1}{3}x_n^{-5}$ yields

x_1	1
x_2	1.16666666666667
x_3	$1.12644367756820$9
x_4	$1.12249706748962$3
x_5	$1.12246205104054$28
x_6	$1.12246204830937$3

- Thus $\sqrt[6]{2} \approx 1.12246205$ to eight decimal places.

A Python Program

IPython: home/goetz

```
Python 3.7.6 (default, Jan  8 2020, 19:59:22)
Type 'copyright', 'credits' or 'license' for more information
IPython 7.19.0 -- An enhanced Interactive Python. Type '?' for help.

In [1]: def f(x): return x**6 - 2

In [2]: def f1(x): return 6*x**5

In [3]: def next(x): return x - f(x)/f1(x)

In [4]: x1 = 1

In [5]: x2 = next(x1); print(x2)
1.1666666666666667

In [6]: x3 = next(x2); print(x3)
1.126443677568209

In [7]: x4 = next(x3); print(x4)
1.122497067489623

In [8]: x5 = next(x4); print(x5)
1.1224620510405428

In [9]: x6 = next(x5); print(x6)
1.122462048309373

In [10]: 
```

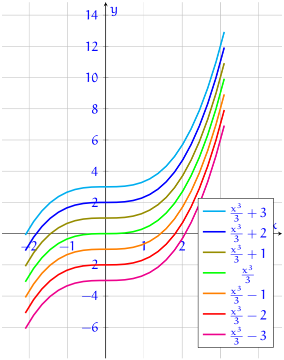
Antiderivatives.

- Can you tell the **position** of a moving particle from its **velocity**?

- A function F is called an **antiderivative** of a function f on an interval I, of $F'(x) = f(x)$ for all $x \in I$.

- An antiderivative of $f(x) = x^2$ is easy to find with the **Power Rule** in mind: $F(x) = \frac{1}{3}x^3$ is one. And $G(x) = \frac{1}{3}x^3 + 17$ is another one.

- As a consequence of the **MVT**, if F and G are any two antiderivatives of f then $G(x) - F(x) = C$ is constant.



A Table of Antiderivatives.

- Every **Differentiation Formula**, when read backwards, gives a **Formula for Antiderivatives**. (Notation: $F' = f$, $G' = g$.)

Function	Antiderivative	Function	Antiderivative
$c\,f(x)$	$c\,F(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\sec x \tan x$	$\sec x$
e^x	e^x	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
b^x	$\dfrac{b^x}{\ln b}$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$

- The antiderivative of a sum is the sum of the antiderivatives . . .

Example.

- A particle moves in a straight line with **position function** $s(t)$.
- Its **velocity** is $v(t) = s'(t)$: $s(t)$ is an **antiderivative** of $v(t)$.
- Its **acceleration** is $a(t) = v'(t)$: $v(t)$ is an **antiderivative** of $a(t)$.

- **Example.** A particle moves in a straight line with acceleration $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.
- **Solution:** The antiderivatives of $v'(t) = a(t) = 6t + 4$ are of the form

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

and $v(0) = -6$ implies $C = -6$, i.e., $v(t) = 3t^2 + 4t - 6$.
- The antiderivatives of $s'(t) = v(t) = 3t^2 + 4t - 6$ are of the form

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

and $s(0) = 9$ implies $D = 9$, i.e., $s(t) = t^3 + 2t^2 - 6t + 9$.

Exercises.

1. Find two numbers whose sum is 23 and whose product is as large as possible by
   (a) making a table of all possible pairs of positive integers and their product,
   (b) using Calculus.

   Compare the two answers you get.

One	Two	Product
1	22	22
2	21	42
⋮	⋮	⋮

2. A farmer wants to fence in an area of $15\,000$ m^2 in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?
3. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.
4. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Exercises.

5. Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the root of the given equation. (Give your answer to four decimal places.)
   (a) $2x^3 - 3x^2 + 2 = 0, \quad x_1 = -1.$
   (b) $\frac{2}{x} - x^2 + 1 = 0, \quad x_1 = 2.$
   (c) $x^7 + 4 = 0, \quad x_1 = -1.$
6. Apply Newton's method to the equation $x^2 - a = 0$ to derive the Babylonian Square Root Algorithm for computing $\sqrt{a}$:
$$x_{n+1} = \frac{1}{2}(x_n + a/x_n).$$
   Then compute $\sqrt{1000}$ correct to six decimal places.
7. Use Newton's method to solve
$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0.$$
8. Find the antiderivative F of f that satisfies the given condition. Check your answer by differentiation.
   (a) $f(x) = 5x^4 - 2x^5, \quad F(0) = 4.$
   (b) $f(x) = 4 - 3(1+x^2)^{-1}, \quad F(1) = 0.$