Week 9: Differentiation Techniques

MA161/MA1161: Semester 1 Calculus.

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Review: Differentiation Rules.

(1) **Power Rule**: $(x^a)' = a x^{a-1}$, for $a \in \mathbb{R}$.

(2) **Constant Multiple Rule**: $(c\,f)' = c\,f'$, for any constant $c \in \mathbb{R}$.

(3) **Sum Rule**: $(f+g)' = f' + g'$.

(4) **Difference Rule**: $(f-g)' = f' - g'$.

(5) **Exponential function**: $(e^x)' = e^x$.

(6) **Product Rule**: $(f\,g)' = f\,g' + g\,f'$.

(7) **Quotient Rule**: $\left(\dfrac{f}{g}\right)' = \dfrac{g\,f' - f\,g'}{g^2}$.

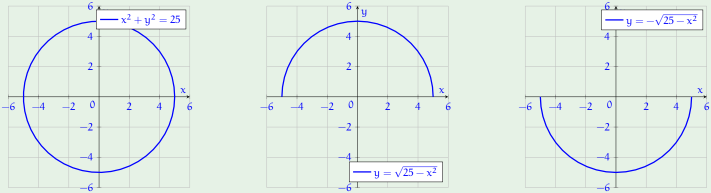
(8) **Trigonometric functions**:
$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$.

(9) **Chain Rule**: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

Implicitly Defined Functions.

- So far, all our functions have expressed one variable **explicitly** in terms of another, like $y = \sqrt{x^2+1}$, or $y = x \sin(x)$.
- We say that a function f is **implicitly defined** by an **equation** like
$$x^2 + y^2 = 25$$
if $x^2 + f(x)^2 = 25$ for all x in the domain of f.
- Sometimes it is possible to **solve** the equation for y as one (or more) explicit functions of x.

- **Example.** Solving $x^2 + y^2 = 25$ for y gives **two functions** $f(x) = \sqrt{25 - x^2}$ and $f(x) = -\sqrt{25 - x^2}$.



Implicit Differentiation.

- The **Folium of Descartes** is a curve defined by the equation

$$x^3 + y^3 = 6xy.$$

which cannot easily be solved for x or y.

- Luckily, we **don't need to solve** a curve's equation for y in order to determine the slope of a tangent to the curve.
- We can use the method of **implicit differentiation** instead.

- **Implicit Differentiation.** Given the equation of a curve.

1. **Differentiate both sides** of the equation wrt. x.
2. **Solve** the resulting equation **for the derivative** y'.

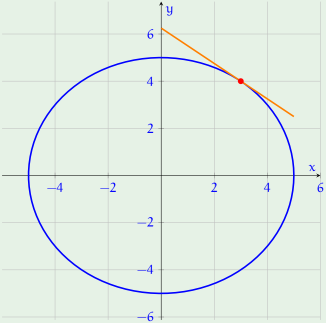
- This process requires a careful use of the **Chain Rule**.

Tangents on a Circle.

- **Example.** Find the equation of the tangent to the circle

$$x^2 + y^2 = 25$$

at the point $(3, 4)$.



- **Solution:**

1. Differentiate both sides:

$\frac{d}{dx}25 = 0$, and

$$\frac{d}{dx}(x^2+y^2) \overset{(3)}{=} \frac{d}{dx}x^2 + \frac{d}{dx}y^2 \overset{(1,9)}{=} 2x + \frac{d}{dy}y^2 \cdot \frac{d}{dx}y \overset{(1)}{=} 2x + 2y \cdot \frac{dy}{dx},$$

using the **Chain Rule** for $\frac{d}{dx}y^2$.

2. Solve $x + y \cdot \frac{dy}{dx} = 0$ for $\frac{dy}{dx}$:

$$y \cdot \frac{dy}{dx} = -x \implies \frac{dy}{dx} = -\frac{x}{y}.$$

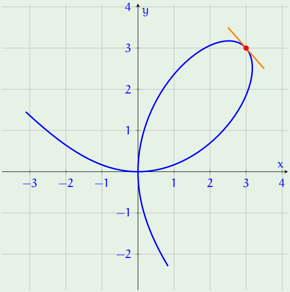
- The slope at $(3, 4)$ is $m = -\frac{3}{4}$.
- Equation of the tangent:

$$y = -\frac{3}{4}(x-3) + 4,$$

using the **point-slope formula** for a line, $y - y_0 = m(x - x_0)$.

The Folium of Descartes.

- **Example.** Find the equation of the tangent to the curve
$$x^3 + y^3 = 6xy$$
at $(3, 3)$.



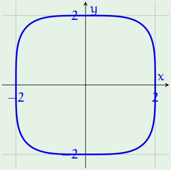
- **Solution:**

1. Differentiate $x^3 + y^3 = 6xy$ wrt. x:
$$\frac{d}{dx}6xy \overset{(6)}{=} 6\left(x\frac{d}{dx}y + y\frac{d}{dx}x\right) = 6(xy' + y)$$
and
$$\frac{d}{dx}(x^3 + y^3) \overset{(3,9)}{=} \frac{d}{dx}x^3 + \frac{d}{dy}y^3\frac{d}{dx}y \overset{(1)}{=} 3x^2 + 3y^2 \cdot y'$$
2. Solve $x^2 + y^2 \cdot y' = 2(xy' + y)$ for y':
$$(y^2 - 2x)y' = 2y - x^2 \implies y' = \frac{2y - x^2}{y^2 - 2x}.$$

- The slope at $(3, 3)$ is $m = \frac{2\cdot 3 - 3^2}{3^2 - 2\cdot 3} = -1$.
- Equation of the tangent:
$$y = -(x - 3) + 3 = 6 - x.$$
- (Recall: $y - y_0 = m(x - x_0)$).

Implicit Second Derivatives.

- We can also find the **second derivative** of an implicit function.



- **Example.** Find y'' if $x^4 + y^4 = 16$.
- **Solution:** Differentiate wrt. x:

$$4x^3 + 4y^3 y' = 0,$$

using the **Power Rule** and the **Chain Rule**.

- Solving for y' gives: $y' = -x^3/y^3$.
- Differentiate again wrt. x, using the **Quotient Rule**:

$$y'' \overset{(2)}{=} -\left(\frac{x^3}{y^3}\right)' \overset{(7)}{=} -\frac{y^3(x^3)' - x^3(y^3)'}{(y^3)^2} \overset{(1,9)}{=} -\frac{3x^2y^3 - 3x^3y^2y'}{y^6}.$$

- Cancelling y^2 and substituting $y' = -x^3/y^3$,

$$y'' = -\frac{3x^2y^2}{y^6}(y - xy') = -\frac{3x^2}{y^4}\left(y + \frac{x^4}{y^3}\right) = -\frac{3x^2}{y^4} \cdot \frac{y^4 + x^4}{y^3},$$

and as $x^4 + y^4 = 16$,

$$y'' = -48x^2/y^7.$$

Derived Inverse Functions.

- **Recall:** a function g is the **inverse function** f^{-1} of a function f, if
$$g(x) = y \Leftrightarrow f(y) = x.$$

- If $y = g(x)$ is inverse to $x = f(y)$ then $y' = \frac{1}{f'(y)}$.

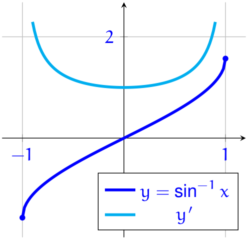
- **Proof.** **Implicit Differentiation** of $x = f(y)$ wrt. x yields
$$1 = \frac{d}{dx}x = \frac{d}{dx}f(y) = \frac{d}{dy}f(y) \cdot \frac{dy}{dx} = f'(y) \cdot y',$$
by the **Chain Rule**. Solving for y' gives $y' = \frac{1}{f'(y)}$. □

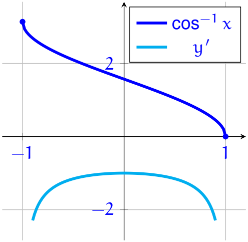
- **Example.** $g(x) = \sqrt{x}$ is the inverse of $f(y) = y^2$. Find $g'(x)$.
- **Solution:** Set $y = \sqrt{x}$. Then $x = y^2$ and $y' = \frac{1}{(y^2)'} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$.
- **Or,** $x = y^2 \implies 1 = \frac{d}{dx}x = \frac{d}{dx}y^2 = 2y \cdot y' \implies y' = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$.

Derived Inverse Trigonometry.

- **Example.** Find y' for $y = \sin^{-1} x$.
- **Solution:** $x = \sin y$ and $y' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$,
  using the **trigonometric identity** $\cos^2 y = 1 - \sin^2 y = 1 - x^2$.

- Similarly, $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$.
- $(\sin^{-1} x)'$ and $(\cos^{-1} x)'$ are **algebraic** functions.

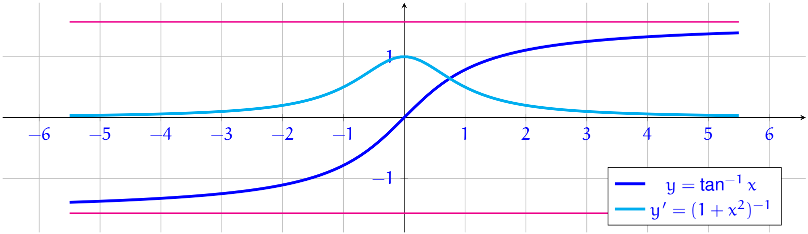




Derived Inverse tan.

- **Example.** Find y' for $y = \tan^{-1} x$.
- **Solution:** $x = \tan y$ and

$$y' = \frac{1}{(\tan y)'} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$



- The derivative of an **inverse trigonometric function** can be a **rational function**.

The Derivative of the Natural Logarithm.

- **Recall:** The **natural logarithm** $\ln x$ is the inverse function of the **exponential function** e^x:

$$\ln x = y \Leftrightarrow e^y = x$$

- **Derivative of the Natural Logarithm:** $(\ln x)' = \frac{1}{x}$.

- **Proof:** Set $y = \ln x$. Then $x = e^y$ and $y' = \frac{1}{(e^y)'} \overset{(5)}{=} \frac{1}{e^y} = \frac{1}{x}$. □

- From $\log_a x = \frac{\ln x}{\ln a}$ it follows that $(\log_a x)' \overset{(2)}{=} \frac{1}{\ln a}(\ln x)' = \frac{1}{(\ln a)x}$.

- **Example.** $(\log_2 x)' = \frac{1}{(\ln 2)x} \approx \frac{1.4427}{x}$.

Log and Chain.

- **Example.** Differentiate $y = \ln(x^3 + 1)$ (wrt. x).
- **Solution:** For the **Chain Rule**, let $u = x^3 + 1$. Then $y = \ln u$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 3x^2 = \frac{3x^2}{x^3 + 1}.$$

- In general, the **Chain Rule** gives

$$\frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u) \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx}, \text{ or } (\ln g(x))' = \frac{g'(x)}{g(x)}.$$

- **Example.** Find $\frac{d}{dx} \ln(\sin x)$.
- **Solution:** $(\ln(\sin x))' = \dfrac{(\sin x)'}{\sin x} = \dfrac{\cos x}{\sin x} = \cot x$.

Differentiating Logarithmic Functions.

- Here, the logarithm is the inner function.

- **Example.** Differentiate $f(x) = \sqrt{\ln x}$.
- **Solution:** By the **Chain Rule**,

$$f'(x) = \frac{1}{2\sqrt{\ln x}} \cdot (\ln x)' = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}.$$

- Here, the base is different from e.

- **Example.** Differentiate $f(x) = \log_{10}(2 + \sin x)$.
- **Solution:** Using $(\log_{10} u)' = \frac{1}{\ln 10 \cdot u}$ with $u = 2 + \sin x$,

$$(\log_{10}(2 + \sin x))' = \frac{(2 + \sin x)'}{\ln 10 \cdot (2 + \sin x)} = \frac{\cos x}{\ln 10 \cdot (2 + \sin x)}.$$

Using the Log Laws.

- **Find** $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.
- **Solution 1:** Using the **Quotient Rule**,

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{\sqrt{x-2}}{x+1}\cdot\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$\overset{(7)}{=} \frac{\sqrt{x-2}}{x+1}\cdot\frac{\sqrt{x-2}\cdot 1-\frac{1}{2}(x+1)(x-2)^{-1/2}}{x-2}$$

$$= \frac{(x-2)-\frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}.$$

- **Solution 2:** Using the **Laws of Logarithms**,

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left(\ln(x+1)-\frac{1}{2}\ln(x-2)\right) \overset{(4)}{=} \frac{1}{x+1}-\frac{1}{2(x-2)},$$

the same as above when written over a common denominator.

Logarithmic Differentiation.

- Solving $(\ln g(x))' = \frac{g'(x)}{g(x)}$ for $g'(x)$ yields
$$g'(x) = g(x)\,(\ln g(x))'.$$
- Using this formula as a method to find $g'(x)$ is called **logarithmic differentiation**.

- **Example.** Differentiate $g(x) = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.
- **Solution:** $\ln g(x) = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$.

Thus $(\ln g(x))' = \dfrac{3}{4}\cdot\dfrac{1}{x} + \dfrac{1}{2}\cdot\dfrac{2x}{x^2+1} - 5\cdot\dfrac{3}{3x+2}$

and $g'(x) = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\dfrac{3}{4x} + \dfrac{x}{x^2+1} - \dfrac{15}{3x+2}\right)$.

Example.

- **Logarithmic Differentiation** can be used to find the derivative of a function of the form $f(x)^{g(x)}$.

- **Example.** Differentiate $g(x) = x^{\sqrt{x}}$.
- **Solution 1:** $\ln g(x) = \sqrt{x}\ln x$.

  Product Rule: $(\ln g(x))' = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}} = \frac{2+\ln x}{2\sqrt{x}}$.

  Hence $g'(x) = x^{\sqrt{x}}\cdot\frac{2+\ln x}{2\sqrt{x}}$.
- **Solution 2:** Write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}} = e^{\sqrt{x}\ln x}$. Then

  $$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}e^{\sqrt{x}\ln x} \overset{(9)}{=} e^{\sqrt{x}\ln x}\frac{d}{dx}(\sqrt{x}\ln x) \overset{(6)}{=} x^{\sqrt{x}}\cdot\frac{2+\ln x}{2\sqrt{x}},$$

  as before.

Power Rule Reloaded.

- **Derivative of the Natural Logarithm:** $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for $x \neq 0$.

- **Proof.** $(\ln|x|)' = \frac{|x|'}{|x|} = \begin{cases} \frac{1}{|x|} = \frac{1}{x}, & \text{if } x > 0, \\ \frac{-1}{|x|} = \frac{1}{x}, & \text{if } x < 0. \end{cases}$ □

- Hence $(\ln|g(x)|)' = \frac{g'(x)}{g(x)}$ and $g'(x) = g(x)\,(\ln|g(x)|)'$.

- **Power Rule:** $(x^a)' = a\,x^{a-1}$ for any $a \in \mathbb{R}$.

- **Proof.** Let $y = x^a$ and use **Logarithmic Differentiation**.
- We have $\ln|y| = \ln|x|^a = a\ln|x|$ and so $(\ln|y|)' = a(\ln|x|)' = \frac{a}{x}$.
- Thus $y' = y \cdot \frac{a}{x} = \frac{a}{x}x^a = ax^{a-1}$.

Indeterminate Forms.

- **Remember** (from weeks 5, 6, and 7) **limits** like

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

- The **limit law for quotients** does **not apply** here, since the denominator $(x-1) \to 0$ as $x \to 1$.
- We solved the above limit by cancelling common factors, $\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1$, for $x \neq 1$.
- Such a strategy **does not work** for limits like

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{or} \quad \lim_{x\to 1} \frac{\ln x}{x-1}.$$

- However, if **both** $f(x) \to 0$ **and** $g(x) \to 0$ as $x \to a$, then the limit $\lim_{x\to a} \left(f(x)/g(x)\right)$ is called an **indeterminate form**, written as $\left[\frac{0}{0}\right]$.

- **Differentiation** can be used to solve such indeterminate forms.

L'Hôpital's Rule.

- **L'Hôpital's Rule.** Suppose that f and g are differentiable functions, and that $g'(x) \neq 0$ on an interval near (but not at) a.

  If **both** $f(x) \to 0$ **and** $g(x) \to 0$ as $x \to a$, then
  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

- That is, instead of computing the limit of the quotient of $f(x)$ and $g(x)$, we compute the limit of the quotient of their **derivatives**.

- **Example.** $\lim_{x\to 1} \frac{\ln x}{x-1} \overset{[\frac{0}{0}]}{=} \lim_{x\to 1} \frac{1/x}{1} = 1$, as $(\ln x)' = \frac{1}{x}$, $(x-1)' = 1$.
- **Example.** $\lim_{x\to 0} \frac{\sin x}{x} \overset{[\frac{0}{0}]}{=} \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$.

- **Important:** check if both $f(x), g(x) \to 0$ before applying the rule!

Computing Indeterminate Forms.

- This limit has been computed in week 6, in a different way.

- **Example.** Evaluate $\lim_{x\to -8} \frac{x^2+11x+24}{x+8}$.
- **Solution:** Set $f(x) = x^2 + 11x + 24$ and $g(x) = x + 8$.

  Compute the limits of f and g by direct substitution:

  $f(x) \to f(-8) = 64 - 88 + 24 = 0$ and $g(x) \to g(-8) = -8 + 8 = 0$.

  So $\lim_{x\to -8} \frac{x^2+11x+24}{x+8}$ is an **indeterminate form** of type $[\frac{0}{0}]$.

  Find the derivatives: $f'(x) = 2x + 11$, $g'(x) = 1$.

  Apply **l'Hôpital's Rule**:

  $$\lim_{x\to -8} \frac{f(x)}{g(x)} \overset{[\frac{0}{0}]}{=} \lim_{x\to -8} \frac{f'(x)}{g'(x)} = \lim_{x\to -8} \frac{2x+11}{1} = -16 + 11 = -5.$$

- **Do not confuse L'Hôpital's Rule with the Quotient Rule!!**

More Examples.

- Sometimes **L'Hôpital's Rule** needs to be applied repeatedly.

- **Example.** Noting that $\sec^2 x \to 1$ as $x \to 0$, we get

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} \overset{[\frac{0}{0}]}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \overset{[\frac{0}{0}]}{=} \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$
$$= \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x} \overset{[\frac{0}{0}]}{=} \frac{1}{3} \lim_{x\to 0} \frac{\sec^2 x}{1} = \frac{1}{3}.$$

- There is also a version of **L'Hôpital's Rule** for $f(x), g(x) \to \pm\infty$.

- **Example.** Calculate $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$.
- **Solution:** $\ln x \to \infty$ and $\sqrt{x} \to \infty$ as $x \to \infty$.

  $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ is an **indeterminate form** of type $[\frac{\infty}{\infty}]$.

  L'Hôpital's Rule: $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim\limits_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim\limits_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$

Exercises.

1. Use the Chain Rule to differentiate the following functions.
   (a) $f(x) = \dfrac{\cos(2x)}{x^3}$.
   (b) $f(x) = \ln(\sqrt{x^2+1})$.
   (c) $f(x) = \sin(5x^2)$.
   (d) $f(x) = \sin^2(5x)$.
2. Prove that, if $f(x) = \cos x$ then $f'(x) = -\sin x$, from first principles. [Hint: $\cos(a+b) = \cos a \cos b - \sin a \sin b$.]
3. Use implicit differentiation to show that
   (a) $\dfrac{d}{dx}\cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$.
   (b) $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$.
4. Find the tangents to the curve $x^2 + xy + 2y^2 = 4$ at the points $(1,1)$ and $(2,-1)$.

Exercises.

5. Use, if possible, l'Hôpital's Rule to evaluate the following limits.

   (a) $\lim_{x\to 2} \frac{2x^2 - 1}{x - 1}$.

   (b) $\lim_{x\to -1} \frac{x^3 + 1}{x + 1}$.

   (c) $\lim_{x\to \pi/2} \frac{\cos x}{x - \pi/2}$.