Week 8: Differentiation Methods

MA161/MA1161: Semester 1 Calculus.

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Differentiation Rules.

- The following rules allow us to find derivatives without calling on First Principles. If $y = f(x)$, we use y' for $f'(x) = \frac{d}{dx} f(x)$.

- The **Power Rule** for $(x^n)'$.
- The **Constant Multiple Rule** for $(c\, f(x))'$.
- The **Sum Rule** for $(f(x) + g(x))'$.
- The **Difference Rule** for $(f(x) - g(x))'$.
- A rule for $(e^x)'$.
- The **Product Rule** for $(f(x) \cdot g(x))'$.
- The **Quotient Rule** for $(f(x)/g(x))'$.
- Derivatives of **Trigonometric Functions**.
- The **Chain Rule** for $(f(g(x)))'$.

(0) **Constant Function Rule**. $(c)' = 0$. (Since $\lim_{h\to 0} \frac{c-c}{h} = 0$.)

Power Rule.

- By First Principles: $(x)' = 1$, $(x^2)' = 2x$, $(x^3)' = 3x^2$, $(x^4)' = 4x^3$.

(1) **Power Rule.** $\frac{d}{dx}x^n = nx^{n-1}$, for $n \in \mathbb{N}$.

- **Proof.** Let $f(x) = x^n$. By the **Binomial Theorem**,
$(x+h)^n = x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n$,
whence $f(x+h) - f(x) = h\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}\right)$.
So $\frac{f(x+h)-f(x)}{h} \longrightarrow nx^{n-1}$ as $h \longrightarrow 0$. □
- In fact, this result extends to any real exponent a:

(1*) **Power Rule.** $\frac{d}{dx}x^a = ax^{a-1}$ for any $a \in \mathbb{R}$.

- **Examples.** $\left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2}$; $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2}$.

New Derivatives from Old.

- When a new function f is formed from old functions by **addition**, **subtraction**, or **muliplying by a constant**, the derivative f' can be calculated from the derivatives of the old functions.

(2) **Constant Multiple Rule**. $(cf)' = c\, f'$ if $c \in \mathbb{R}$ is any constant.

- **Proof.** Let $g(x) = c\, f(x)$. Then

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{c\, f(x+h) - c\, f(x)}{h}$$
$$= \lim_{h\to 0} c\, \frac{f(x+h)-f(x)}{h} = c \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = c\, f'(x),$$

by the **Constant Multiple Law** for limits. □

- **Examples.** $(3x^4)' = 3(x^4)' = 3 \cdot 4x^3 = 12x^3$.
- $\frac{d}{dx}(-x) = \frac{d}{dx}((-1)\, x) = -\frac{d}{dx}(x) = -1$.

Sums and Differences.

- The derivative of a sum of functions is the sum of the derivatives.

(3) **Sum Rule:** $(f+g)' = f' + g'$ if both f and g are differentiable.

(4) **Difference Rule:** $(f-g)' = f' - g'$ if both f' and g' exist.

- Together, rules (0) – (4) determine **derivatives of polynomials**.

- **Example.** Differentiate $f(x) = 3x^4 - 5x^2 + 7$.
- **Solution:** $f'(x) = (3x^4 - 5x^2 + 7)' \overset{(3,4)}{=} (3x^4)' - (5x^2)' + (7)'$
$\overset{(0,2)}{=} 3(x^4)' - 5(x^2)' \overset{(1)}{=} 3 \cdot 4x^3 - 5 \cdot 2x = 12x^3 - 10x$.

- **Find** the **acceleration** of an object with $s(t) = 2t^3 - 5t^2 + 3t + 4$.
- **Solution:** $v(t) = s'(t) = (2t^3 - 5t^2 + 3t + 4)' = 6t^2 - 10t + 3$.
$a(t) = v'(t) = (6t^2 - 10t + 3)' = 12t - 10$. So, e.g., $a(2) = 14$.

Derivatives of Exponential Functions.

- Let's try and compute f' for $f(x) = a^x$ from first principles.
- As $a^{x+h} - a^x = a^x a^h - a^x = a^x(a^h - 1)$, we find that

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = f'(0)\, a^x.$$

- The derivative of an exponential function $f(x) = a^x$ is, remarkably, a constant multiple of the function $f(x)$ itself:

- If $f(x) = a^x$ then $f'(x) = f'(0)\, a^x$.

- **Euler's number** $e = 2.71828\ldots$ has the property $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

(5) **Derivative of the Exponential Function:** $(e^x)' = e^x$.

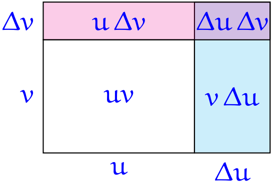
- The rate of change of e^x equals e^x. More on this later . . .

Products of Functions.

- The derivative of a product is **not** the product of the derivatives.

- **Example.** If $f(x) = g(x) = x$ then $f'(x) = g'(x) = 1 = f'(x)\,g'(x)$. But $(fg)(x) = f(x)\,g(x) = x^2$ and $(fg)'(x) = 2x \neq 1$.

- Let's see why. Suppose $u = f(x)$ and $v = g(x)$ are two quantities.
- The product uv is the size of a rectangle.
- If x changes by Δx then $\Delta u = f(x + \Delta x) - f(x)$, and $\Delta v = \dots$
- $(u + \Delta u)(v + \Delta v)$ is the larger rectangle.
- $\Delta(uv) = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v$.



- $$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x} \xrightarrow{\Delta x \to 0} u\frac{dv}{dx} + v\frac{du}{dx} + 0 \cdot \frac{dv}{dx}.$$

- Thus, in Leibniz notation, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

The Product Rule.

(6) **Product Rule:** If both f and g are differentiable functions then

$$(f\,g)' = f\,g' + g\,f'.$$

- In words, the **derivative of a product** is a **sum** of mixed terms, each the product of one function and the derivative of the other.

- **Example.** For $f(x) = x\,e^x$, find $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$.
- **Solution:** $f'(x) = (x\,e^x)' = x\,(e^x)' + e^x\,(x)'$
  $= x\,e^x + e^x \cdot 1 = (x+1)\,e^x$
- $f''(x) = ((x+1)\,e^x)' = (x+1)\,(e^x)' + e^x\,(x+1)'$
  $= (x+1)\,e^x + e^x \cdot 1 = (x+2)\,e^x$
- $f'''(x) = ((x+2)\,e^x)' = \cdots = (x+3)\,e^x$,
- $\ldots$
- $f^{(n)}(x) = (x+n)\,e^x$.

Applications of the Product Rule.

- **Example.** If $f(x) = \sqrt{x}\, g(x)$, $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.
- **Solution:** $f'(x) = \left(\sqrt{x}\, g(x)\right)' = \sqrt{x}\, g'(x) + \frac{g(x)}{2\sqrt{x}}$.

  Hence $f'(4) = \sqrt{4}\, g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{4} = \frac{13}{2}$.

- Sometimes there are alternatives to using the Product Rule.

- **Example.** Differentiate $f(x) = (2x^3 + 1)(3x - 2)$.
- **Solution 1:** $f'(x) = uv$, where $u = 2x^3 + 1$ and $v = 3x - 2$.

  Thus $u' = 6\,x^2$ and $v' = 3$, and
  $(uv)' = uv' + vu' = 3(2x^3 + 1) + 6x^2(3x - 2) = \underline{24x^3 - 12x^2 + 3}$.
- **Solution 2:** Expanding the product, $f(x) = 6x^4 - 4x^3 + 3x - 2$.

  Thus $f'(x) = \underline{24x^3 - 12x^2 + 3}$, by Rules (0) – (4).

Quotients of Functions.

- We can derive a rule for the derivative of a quotient in a similar way as we found the **Product Rule**.
- If $u = f(x)$ and $v = g(x)$, and if x, u, v change by Δx, Δu, Δv respectively, then the change in the quotient u/v is

$$\Delta\left(\frac{u}{v}\right) = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} = \frac{(u+\Delta u)\,v - u\,(v+\Delta v)}{v\,(v+\Delta v)} = \frac{v\,\Delta u - u\,\Delta v}{v\,(v+\Delta v)}$$

- So $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \lim_{\Delta x\to 0}\dfrac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x\to 0}\dfrac{v\,\frac{\Delta u}{\Delta x} - u\,\frac{\Delta v}{\Delta x}}{v\,(v+\Delta v)} = \dfrac{v\frac{du}{dx} - u\frac{v}{dx}}{v^2}$, by a careful application of the **Limit Laws**.

- $$\lim_{\Delta x\to 0}\frac{v\,\frac{\Delta u}{\Delta x} - u\,\frac{\Delta v}{\Delta x}}{v\,(v+\Delta v)} = \cdots = \frac{v\,\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} - u\,\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}}{v\,\lim_{\Delta x\to 0}(v+\Delta v)}$$
- Note that here $\Delta v \to 0$ as $\Delta x \to 0$, since $v = g(x)$ is differentiable, hence continuous.

The Quotient Rule.

(7) **Quotient Rule:** If f and g are both differentiable, $g \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{g\,f' - f\,g'}{g^2}.$$

- In words, the **derivative of a quotient** is a **difference** of mixed terms, divided by the **square** of the denominator.

- **Example.** Differentiate $y = e^x/x$.
- **Solution:** $y = f(x)/g(x)$, where $f(x) = e^x$ and $g(x) = x$.
- Thus $f'(x) = e^x$ and $g'(x) = 1$.
- Hence, by the **Quotient Rule**,

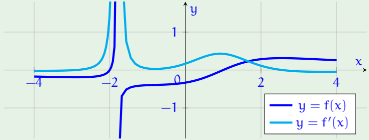
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{g(x)^2} = \frac{x\,e^x - e^x \cdot 1}{x^2} = \frac{x-1}{x^2}e^x.$$

Applications of the Quotient Rule.

- The **Quotient Rule** enables us to differentiate **rational** functions.

- **Example.** Differentiate

$$f(x) = \frac{x^2 + x - 2}{x^3 + 6}.$$

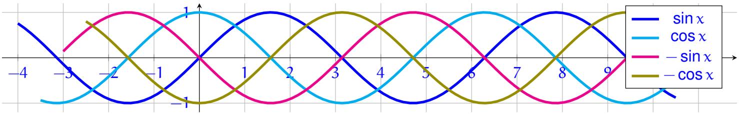


- **Solution:**

$$\begin{aligned} f'(x) &= \frac{(x^3+6)(x^2+x-2)' - (x^2+x-2)(x^3+6)'}{(x^3+6)^2} \\ &= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2} \\ &= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2} \\ &= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2} \end{aligned}$$

Derivatives of Trigonometric Functions.

- Recall: The graphs of $\pm\sin x$ and $\pm\cos x$ look like this:



- Given that the derivative of $f(x)$ at $x = a$ is the slope of the tangent to $f(x)$ at a, it's tempting to guess $(\sin x)'$ and $(\cos x)'$.

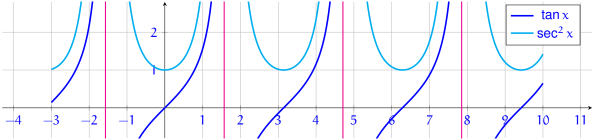
(8) **Derivatives of Trigonometric Functions:**

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\tan x = \sec^2 x.$$

- **Proof.** The derivative of $\tan x = \frac{\sin x}{\cos x}$ uses the **Quotient Rule**:
- $\left(\frac{\sin x}{\cos x}\right)' = \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$ □

Examples

- The graphs of $\tan x$ and $(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$.



- **Example.** Differentiate $f(x) = x^2 \sin x$.
- **Solution:** Using the **Product Rule**,

  $f'(x) = x^2(\sin x)' + (\sin x)(x^2)' = x^2 \cos x + 2x \sin x$.

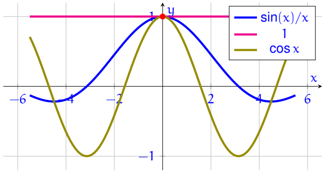
- **Example.** Differentiate $f(x) = \sin(x)\cos(x)$.
- **Solution:** Using the **Product Rule**,

  $f'(x) = (\sin x)(\cos x)' + (\cos x)(\sin x)' = \cos^2 x - \sin^2 x$.

Some Limits.

- The following limits will be useful for differentiating $\sin x$.

- $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$,
- $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$.



- $\sin \theta \leqslant \theta \leqslant \tan \theta$ if $\theta \in (0, \frac{\pi}{2})$.
- $\sin \theta \leqslant \theta \implies \frac{\sin \theta}{\theta} \leqslant 1$. And $\theta \leqslant \tan \theta = \frac{\sin \theta}{\cos \theta} \implies \cos \theta \leqslant \frac{\sin \theta}{\theta}$.

  Squeeze: $\lim_{\theta \to 0} \cos x = \lim_{\theta \to 0} 1 = 1 \implies \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$. □
- $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \frac{\cos^2 \theta - 1}{\theta (\cos \theta + 1)} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} = 1 \cdot \frac{0}{1+1} = 0$. □

- **Example.** Find $\lim_{x \to 0} \frac{\sin 7x}{4x}$.
- **Solution:** $\lim_{x \to 0} \frac{\sin 7x}{4x} = \lim_{x \to 0} \frac{7}{4} \frac{\sin 7x}{7x} = \frac{7}{4} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{7}{4} \cdot 1 = \frac{7}{4}$,

  where $\theta = 7x$ and $\theta \to 0$ as $x \to 0$.

Differentiating $\sin(x)$.

- $(\sin x)' = \cos x$.

- Use the **Addition Formula** $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

- $$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
  $$= \lim_{h\to 0}\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
  $$= \lim_{h\to 0}\left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}\right)$$
  $$= \sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$
  $$= \sin x \cdot 0 + \cos x \cdot 1$$
  $$= \cos x. \qquad \square$$

- In a similar way, you can show that $(\cos x)' = -\sin x$.

Cotangent, Secant, Cosecant

- The **multiplicative inverses** of $\sin x$, $\cos x$ and $\tan x$ are:

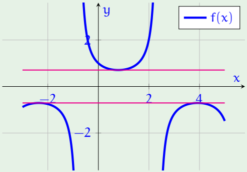
$$\cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

- Their derivatives can be found by using the **Quotient Rule** (try!):

$$(\cot x)' = -\csc^2 x,$$
$$(\sec x)' = \sec x \tan x,$$
$$(\csc x)' = -\csc x \cot x.$$

- **Example.** Differentiate $f(x) = \frac{\sec x}{1+\tan x}$.
- **Solution:** By the **Quotient Rule**,

$$f'(x) = \frac{(1+\tan x)(\sec x)' - \sec x(1+\tan x)'}{(1+\tan x)^2}$$
$$= \frac{(1+\tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$
$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$



using the identity $\tan^2 x + 1 = \sec^2 x$.

The Chain Rule.

- **Example.** What is the derivative of $F(x) = \sqrt{x^2+1}$?
- Note that $F = f \circ g$ is a **composite**: $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$.
- If $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$ then $y = F(x) = f(g(x))$.

- The derivative of $f \circ g$ is the **product** of the derivatives of f and g.

(9) **Chain Rule:** If $y = f(u)$ and $u = g(x)$ are both differentiable then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- Or, in **prime notation**: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

- **Solution 1:** $\frac{d}{dx}F(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$.
- **Solution 2:** $F'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$.

The Chain Rule: Examples.

- Recall: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. Or $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.
- From **Outside** to **Inside**: Differentiate the outer function f (at the inner function g) then multiply by derivative of inner function g.

- **Example.** Differentiate (a) $y = \sin x^2$ and (b) $y = \sin^2 x$.
- **Solution:** $(x^2)' = 2x$ and $(\sin x)' = \cos x$.

(a) The outer function is $\sin u$, the inner function is x^2:

$$(\sin x^2)' = \sin'(x^2) \cdot (x^2)' = \cos(x^2) \cdot 2x = 2x \cos x^2$$

(b) $\sin^2 x = (\sin x)^2$: outer function f is u^2, inner function g is $\sin x$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u \cdot \cos x = 2 \sin x \cos x,$$

where

$$2 \sin x \cos x = \sin 2x$$

by the **Addition Formula** for $\sin(a + b)$.

Power and Chain

- If $y = u^n$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}$.

- If $f(x) = g(x)^n$ then $f'(x) = ng(x)^{n-1} \cdot g'(x)$.

- **Example.** Differentiate $f(x) = (x^3 - 1)^{100}$.
- **Solution:** With $g(x) = x^3 - 1$ and $n = 100$, we get $f'(x) = 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$.

- **Example.** Find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.
- **Solution:** Combining **Power and Chain** with the **Quotient Rule**, $g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8\left(\frac{t-2}{2t+1}\right)' = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$.

Exponential Chain.

- Earlier, we found that $(a^x)' = \lim_{h\to 0} \frac{a^h - 1}{h} a^x$.
- Also, since $a = e^{\ln a}$, we have $a^x = (e^{\ln a})^x = e^{(\ln a)x}$.
- **Chain Rule**: $(a^x)' = (e^{(\ln a)x})' = e^{(\ln a)x}((\ln a)x)' = a^x \ln a$.

- $(a^x)' = \ln a \cdot a^x$.

- **Example.** $a = 2$:

$$(2^x)' = \ln 2 \cdot 2^x \approx 0.693147 \cdot 2^x.$$

- **Example.** $a = e$:

$$(e^x)' = \ln e \cdot e^x = e^x.$$

Longer Chains.

- Suppose that $y = f(u)$, $u = g(x)$ and $x = h(t)$. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt},$$

using the **Chain Rule** twice.

- **Example.** If $f(x) = \sin(\cos(\tan x))$ then working outside to inside,

$$f'(x) = \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot \sec^2 x.$$

- **Example.** Differentiate $y = e^{\sec 3\theta}$.
- **Solution:** with $(e^x)' = e^x$, $(\sec x)' = \sec x \tan x$, $(3x)' = 3$,

$$\begin{aligned}\frac{d}{d\theta} e^{\sec 3\theta} &= e^{\sec 3\theta} \frac{d}{d\theta} \sec 3\theta \\ &= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta} 3\theta \\ &= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta\end{aligned}$$

Derived Yoga.

- $\frac{d}{dx}\big(a\, f(cx+d)+b\big) = a\, c\, f'(cx+d)$.

- **Proof:** Note that $a\, f(cx+d)+b = (B \circ A \circ f \circ D \circ C)(x)$, where $A(x) = ax$, $B(x) = x+b$, $C(x) = cx$, and $D(x) = x+d$.
  Chain Rule: $\big(a\, f(cx+d)+b\big)' =$
  $$\underbrace{B'(a\, f(cx+d))}_{=1} \cdot \underbrace{A'(f(cx+d))}_{=a} \cdot f'(cx+d) \cdot \underbrace{D'(cx)}_{=1} \cdot \underbrace{C'(x)}_{=c} \qquad \square$$
- The special cases $c = -1$, or $a = -1$ yield the following fun fact.

- The derivative of an **even** function is **odd**, and vice versa.

- **Proof:** $\frac{d}{dx} f(-x) = -f'(-x)$, and $\frac{d}{dx}(-f(x)) = -f'(x)$, by **above**.
- If f is **even** then $f(-x) = f(x)$ and thus $-f'(-x) = f'(x)$.
  So $f'(-x) = -f'(x)$ and f' is **odd**.
- If f is **odd** then $f(-x) = -f(x)$ and thus $-f'(-x) = -f'(x)$.
  So $f'(-x) = f'(x)$ and f' is **even**. $\square$

Summary: Differentiation Rules.

1. **Power Rule**: $(x^a)' = ax^{a-1}$, for $a \in \mathbb{R}$.
2. **Constant Multiple Rule**: $(c\,f)' = c\,f'$, for any constant $c \in \mathbb{R}$.
3. **Sum Rule**: $(f+g)' = f' + g'$.
4. **Difference Rule**: $(f-g)' = f' - g'$.
5. **Exponential function**: $(e^x)' = e^x$.
6. **Product Rule**: $(f\,g)' = f\,g' + g\,f'$.
7. **Quotient Rule**: $\left(\frac{f}{g}\right)' = \frac{g\,f' - f\,g'}{g^2}$.
8. **Trigonometric functions**:
   $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$.
9. **Chain Rule**: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

Exercises.

1. Show that $\frac{d}{dx}x^4 = 4x^3$ from first principles.
2. Differentiate the following functions.
   (i) $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$.
   (ii) $f(x) = \dfrac{1+x}{1-x}$.
3. (MA161 Exam, Summer 2011/2012) Consider the piecewise defined function

$$f(x) = \begin{cases} x^2, & \text{if } x \leqslant 6, \\ qx + r, & \text{if } x > 6. \end{cases}$$

   For which values of q and r are both $f(x)$ and $f'(x)$ continuous at $x = 6$?
4. Use the limit laws to show that $(f(x) + g(x))' = f'(x) + g'(x)$.

   [Hint. Complete and justify $\lim_{h \to 0} \frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} = \cdots = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$.]

Exercises.

5. Find the 27th derivative of $\cos(x)$.
6. Differentiate $f(x) = e^{kx}$ for a constant $k \in \mathbb{R}$.
7. Differentiate the following functions.
   (a) $f(x) = 4x^3 + e^{4x}$.
   (b) $f(x) = x^2 \cos(x)$.
   (c) $f(x) = \cos(x)/x^3$.
   (d) $f(x) = \sqrt[3]{x^2 + 2x + 1}$.
   (e) $f(x) = (x^2 + 1)^6$.
8. Find the tangents to the function $f(x) = x^2 + x - 6$ at the points $x = -4$ and $x = 2$.
9. Find the equation for the tangent to $f(x) = e^{-x} - \sqrt{x+1} + 1$ at $x = 0$.
10. It is possible to express $f(x) = e^x$ as the infinite series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n + \cdots ,$$

where $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.
Use this expression to show that $\frac{d}{dx}e^x = e^x$.