Week 7: The Derivative

MA161/MA1161: Semester 1 Calculus.

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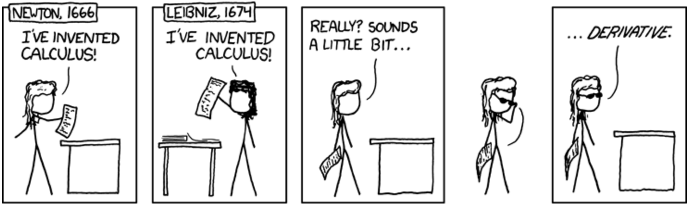


Tangents Revisited.

- Using **limits**, we can now give a precise definition of a **tangent**.

- The **tangent line** to a curve $y = f(x)$ at the point $P = (a, f(a))$ is the straight line passing through P with **slope**

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

- **Find** an equation of the tangent to the parabola $y = x^2$ at $(1, 1)$.
- **Solution:** With $a = 1$ and $f(x) = x^2$, we have

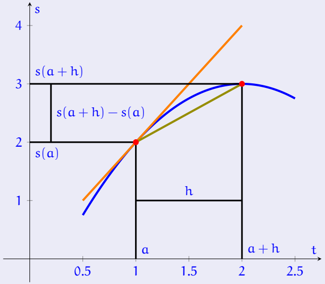
$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

- Hence the **point-slope form** of the equation of a line yields $y - 1 = 2(x - 1)$ as the **equation of the tangent**, or simply

$$y = 2x - 1.$$

Velocity.

- An **object** moves along a **straight line**.
- Its **displacement** s from the origin at time t is described by a function $s(t)$.
- For any $h > 0$, in the **time interval** from a to $a + h$ the object is displaced by $s(a + h) - s(a)$.



- Its **average velocity** over this interval is

$$\frac{\text{displacement}}{\text{time}} = \frac{s(a+h) - s(a)}{h}.$$

- The **instantaneous velocity** $v(a)$ at time a is the **limit**

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

Free Fallin'

- **Example.** A ball is dropped from 450 m above ground.

(a) What is its velocity after 5 seconds?

(b) How fast is the ball when it hits the ground?

- **Solution:** By **Galileo's equation of motion**, $s(t) = 4.9\,t^2$.

(a) Thus at $t = 5$,

$$v(5) = \lim_{h\to 0} \frac{s(5+h) - s(5)}{h} = 4.9 \lim_{h\to 0} \frac{(5+h)^2 - 5^2}{h}$$

$$= 4.9 \lim_{h\to 0} \frac{25 + 10h + h^2 - 25}{h} = 4.9 \lim_{h\to 0} \frac{10h + h^2}{h}$$

$$= 4.9 \lim_{h\to 0} (10 + h) = 49 \text{ m/s.}$$

(b) When $s(t) = 450$, we have $4.9\,t^2 = 450 \implies t = \sqrt{\frac{450}{4.9}} \approx 9.6$ s.

- $v(t) = 4.9 \lim_{h\to 0} \frac{(t+h)^2 - t^2}{h} = 4.9 \lim_{h\to 0} (2t + h) = 4.9 \cdot 2t \approx 94$ m/s.

The Derivative.

- A limit of this form, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, is called a **derivative**.

- The **derivative** $f'(a)$ **of a function** f **at a point** a is
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$
provided this limit exists.

- If $b = a + h$ then $h = b - a$ and $h \to 0$ in the same way as $b \to a$.
- Thus an **equivalent formula** for $f'(a)$ is
$$f'(a) = \lim_{b\to a} \frac{f(b)-f(a)}{b-a},$$

- **Example.** If $f(x) = x^2$ then $f'(1) = \lim_{b\to 1} \frac{b^2-1}{b-1} = \lim_{b\to 1}(b+1) = 2$.
- Or, $f'(1) = \lim_{h\to 0} \frac{(1+h)^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2$.

Equation of a Tangent

- Using the **derivative**, we can now give the equation of a tangent.

- The **equation of the tangent line** to $y = f(x)$ at $(a, f(a))$ is
$$y = f'(a)(x - a) + f(a)$$

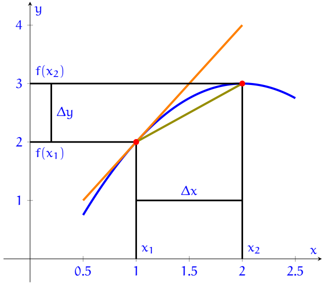
- **Example.** Find the derivative of $f(x) = x^2 - 8x + 9$ at a.
- **Solution:** With $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, we have
$$f'(a) = \lim_{h\to 0} \frac{\left((a+h)^2 - 8(a+h) + 9\right) - \left(a^2 - 8a + 9\right)}{h} = \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h},$$
hence $f'(a) = \lim_{h\to 0} (2a + h - 8) = 2a - 8$.
- **Example.** Now, find the equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.
- **Solution:** As $f'(3) = 2 \cdot 3 - 8 = -2$, the equation of the tangent is
$$y = (-2)(x - 3) - 6, \text{ or simply } y = -2x.$$

Rates of Change

- Suppose a quantity y depends on another quantity x: $y = f(x)$.
- If x changes from x_1 to x_2 then the **change in** x is $\Delta x = x_2 - x_1$.
- The **corresponding change in** y is $\Delta y = f(x_2) - f(x_1)$.
- The **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change** of y with respect to x over $[x_1, x_2]$.



- The **(instantaneous) rate of change** of y wrt. x at x_1 is the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

- Thus $f'(a)$ is the **rate of change** of $y = f(x)$ wrt. x at $x = a$.

Example: Production Costs.

- A manufacturer produces **bolts of a fabric** with a fixed width. The **cost** of producing x meters of this fabric is $C = f(x)$ euro.

(a) What is the meaning of the derivative $f'(a)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ of $f'(500)$? Or $f'(5000)$?

- **Solution:**

(a) $f'(x)$ is the rate of change of C wrt. x, i.e., the rate of change of the **production cost** relative to the number of meters produced: the **marginal cost**.

Its units are the same as the units of $\Delta C/\Delta x$: **euro per meter**.

(b) $f'(1000) = 9$ means that after 1000 meters, production cost **increases at a rate** of 9 €/m: the 1001st meter costs €9.

(c) Presumably, $f'(500) < f'(50)$ because of **economies of scale**.

Possibly, $f'(5000) > f'(500)$ as **large-scale is inefficient** ...

Example: Covid-19 Cases

- This table lists the **cumulative total number** of Covid-19 case in **Co. Galway**, at the end of the week for the five weeks leading up to level 5 restrictions. Source: Wikipedia

Week t	# Cases
38	564
39	618
40	723
41	859
42	1123
43	1602

- We regard this as a **sample of values** of a function $C(t)$.
- **Estimate** and **interpret** the value of $C'(40)$.
- **Solution:** $C'(40)$ is the **rate of change** of C wrt. t at $t = 40$:

$$C'(40) = \lim_{t \to 40} \frac{C(t) - C(40)}{t - 40}$$

t	Interval	$\Delta C/\Delta t$
38	$[38, 40]$	79.5
39	$[39, 40]$	105.0
41	$[40, 41]$	136.0
42	$[40, 42]$	200.0

- We can compute and tabulate values of the **difference quotient** $\Delta C/\Delta t$ for t close to 40.
- Conclusion: $105.0 \leqslant C'(40) \leqslant 136.0$

- **Read** Section 3.7 for Rates of Change in the Natural Sciences.

Derivative as a Function

- We may regard the process of finding $f'(a)$ for a as a function.

- Suppose $f\colon D \to \mathbb{R}$ is a function. The function f', defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists, is called the **derivative** of f.

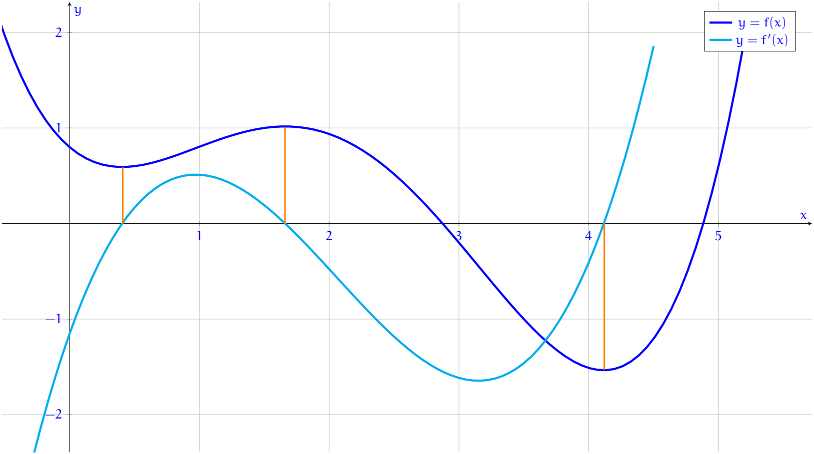
- The **domain** of f' is the set $\{x \in D \mid f'(x) \text{ exists}\}$ which may be smaller than the domain D of f.

- **Find** $f'(x)$ if $f(x) = x^2$.
- **Solution:** Using $(x+h)^2 = x^2 + 2xh + h^2$,

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x + h) = 2x.$$

A Function and its Derivative

- You can produce a sketch of the graph of f' from the graph of f.



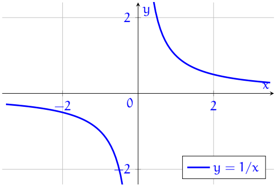
- **Note** how $f'(x) < 0$ when $f(x)$ is **decreasing**, how $f'(x) > 0$ ...
- **Note** how $f'(x) = 0$ when the tangent at $f(x)$ is **horizontal**.

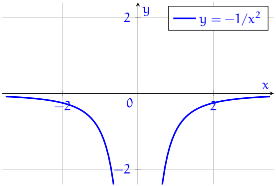
First Principles

- Using the definition to find the derivative f' of a function f is called **differentiation from first principles**.

- **Find** the derivative of $f(x) = \frac{1}{x}$ from first principles.
- **Solution:** Using $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$,

$$f'(x) = \lim_{h\to 0} \left(\frac{1}{x+h} - \frac{1}{x}\right)/h = \lim_{h\to 0} \frac{x-(x+h)}{(x+h)xh} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$



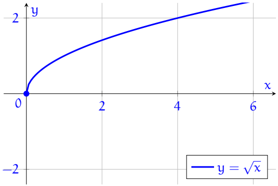


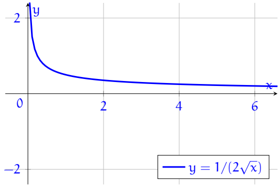
Example.

- **Find** the derivative of $f(x) = \sqrt{x}$ from first principles.
- **Solution: Rationalizing the numerator**, we get

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

- Thus $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$, if $x > 0$.





Differentiable Implies Continuous

- A function f is **differentiable** at a if $f'(a)$ exists.
- The function f is differentiable on an open interval I, if it is differentiable at every $a \in I$.

- Both **continuity** and **differentiability** are desirable properties for a function f to have. They are related as follows.

- If a function f is **differentiable** at a then f is **continuous** at a.

- **Proof:** Since $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exits, the **Product Law** yields

$$\left(\lim_{x \to a} f(x)\right) - f(a) = \lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot (x-a)$$
$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} (x-a) = f'(a) \cdot 0 = 0.$$

Hence $\lim_{x \to a} f(x) = f(a)$ and f is continuous at a. □

How Can a Function Not Be Differentiable?

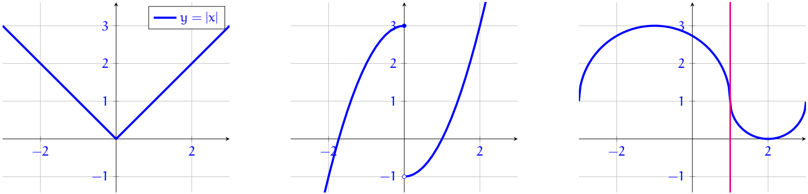
- **Example.** Where is $f(x) = |x|$ differentiable?
- **Solution:** If $x > 0$ then $|x| = x$ and $|x+h| = x+h$ for small h:

$$f'(x) = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1.$$

- Similarly, if $x < 0$ then $|x| = -x$ and $|x+h| = -(x+h)$ for small h:

$$f'(x) = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

- For $x = 0$ we have $\lim_{h\to 0+} \frac{|h|}{h} = \lim_{h\to 0+} \frac{h}{h} = 1$ and $\lim_{h\to 0-} \frac{|h|}{h} = -1$, whence $f'(0) = \lim_{h\to 0} \frac{|h|}{h}$ **does not exist**.



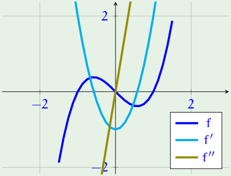
Second Derivative

- The derivative f' of a function f may have its own derivative.

- The **second derivative** of a differentiable function f is the derivative of the derivative, denoted by $f'' = (f')'$.

- **Example.** Find $f''(x)$ for $f(x) = x^3 - x$.
- **Solution:** $f'(x) = 3x^2 - 1$ (from First Principles: Show!).

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \cdots = 6x.$$



- $f''(x)$ is the **rate of change** of the **slope** of $y = f(x)$.

- An object moves along a straight line. The **first derivative** of its **position** function $s(t)$ is its **velocity** $v(t) = s'(t)$.
- The **second derivative** is its **acceleration** $a(t) = v'(t) = s''(t)$.

Higher Derivatives

- The **third derivative** of a differentiable function f is the derivative of the second derivative, denoted by $f''' = (f'')'$.

- **Did you know?** The derivative of the acceleration $a(t)$ of a moving object is called its **jerk**: $j(t) = a'(t) = v''(t) = s'''(t)$.

- In general, the n-th derivative of f is denoted by $f^{(n)}$.

- **Notation.** There are **several ways** to write f' if $y = f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx} f(x).$$

- Here, the symbol $\frac{dy}{dx}$ should **not be read as a fraction**.
- The symbol $\frac{d}{dx}$ can be read as a **differentiation operator**.
- $f''(x) = (\frac{d}{dx})^2 y = \frac{d^2y}{dx^2}$. $f'''(x) = \frac{d^3y}{dx^3}$. $f^{(n)}(x) = \frac{d^ny}{dx^n}$.

Exercises.

1. Let $f(x) = 7 - 13x$. Use the definition of the derivative to find $f'(-2)$. What is the derivative f' as a function of x?
2. Let $f(x) = x^3$. Use the definition of that derivative to show that $f'(x) = 3x^2$.
3. Use the trigonometric identity

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

to show that $\frac{d}{dx}\sin(x) = \cos(x)$.

[Hint. $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.]

4. Use the trigonometric identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

to show that the derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.

5. What is the equation of the tangent to $f(x) = x^2 + 4x + 3$ at $x = 1$?