Week 6: Continuity; Horizontal Asymptotes

MA161/MA1161: Semester 1 Calculus.

Prof. Götz Pfeiffer

School of Mathematics, Statistics and Applied Mathematics
NUI Galway

November 02–03, 2020

Calculating Certain Limits.

- Not all limits can be calculated by direct substitution. But still . . .

- $\lim_{x \to 1} \frac{x^2-1}{x-1} = ?$ Algebra: $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ if $x \neq 1$.
- Thus $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} (x+1) = 2$, by direct substitution.

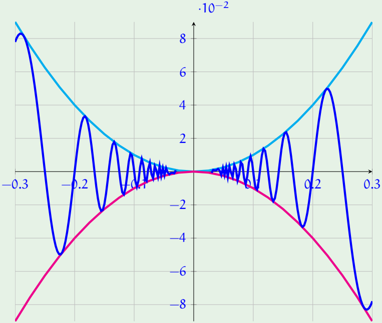
- If $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

- **Evaluate** $\lim_{h \to 0} \frac{(3+h)^2-9}{h}$.
- **Solution:** If $h \neq 0$ then $\frac{(3+h)^2-9}{h} = \frac{9+6h+h^2-9}{h} = \frac{(6+h)h}{h} = 6+h$.
- Thus $\lim_{h \to 0} \frac{(3+h)^2-9}{h} = \lim_{h \to 0} (6+h) = 6$, by direct substitution.

Squeeze.

- **Squeeze Theorem:** If $f(x) \leqslant h(x) \leqslant g(x)$ for all x near a (but $x \neq a$) and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

- **Show that** $\lim_{x\to 0} x^2 \sin \frac{\pi}{x} = 0$.
- **Solution:**
- $|\sin x| \leqslant 1 \implies -1 \leqslant \sin \frac{\pi}{x} \leqslant 1$.
- Hence $-x^2 \leqslant x^2 \sin \frac{\pi}{x} \leqslant x^2$.
- But $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0$.
- By **Squeeze**, $\lim_{x\to 0} x^2 \sin \frac{\pi}{x} = 0$.



Continuity at a Point.

- Often, the limit of a function $f(x)$ as $x \to a$ can be found simply by calculating $f(a)$. We then say that "f is **continuous** at a".

- A function $f\colon D \to \mathbb{R}$ is **continuous** at a point $a \in D$ if

$$\lim_{x\to a} f(x) = f(a).$$

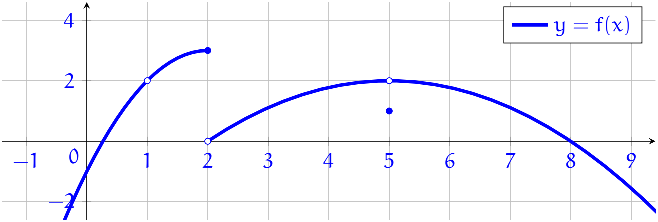
- Note that this definition of continuity requires **three things**:

1. $f(a)$ is defined (that is, a is in the domain of f),
2. the limit $\lim_{x\to a} f(x)$ exists,
3. and $\lim_{x\to a} f(x) = f(a)$.

- Physical phenomena are usually continuous.
- A function f is continuous at a if $f(x) \to f(a)$ as $x \to a$: small changes in x cause only small changes in $f(x)$.

Discontinuity in Graphs.

- If $f(x)$ is defined near a but **not continuous** at a we say that f has a **discontinuity** at a.



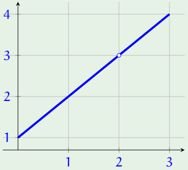
- **Example.** The graph of a function f is shown above. **Where** is f **discontinuous**? And **why**?
- At $a = 1$, $\lim_{x \to a} f(x)$ exists but $f(a)$ **is not defined**.
- At $a = 2$, $f(a)$ is defined but $\lim_{x \to a} f(x)$ **does not exist**.
- At $a = 5$, both $f(a)$ and $\lim_{x \to a} f(x)$ do exist, but **they are distinct**.

Discontinuity in Formulas.

- The function

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

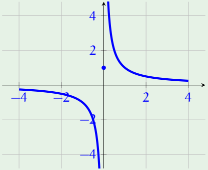
is not continuous at 2, as $f(2)$ **is not defined**.



- The function

$$f(x) = \begin{cases} x^{-1}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

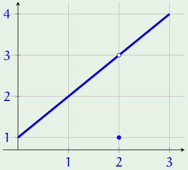
is not continuous at 0, as $\lim_{x \to 0} f(x)$ **does not exist**.



- The function

$$f(x) = \begin{cases} x + 1, & x \neq 2, \\ 1, & x = 2, \end{cases}$$

is not continuous at 2, as $\lim_{x \to 2} f(x) = 3$ **differs from** $f(x) = 1$.



Direct Substitution Reloaded.

- If the functions f and g are continuous at a, and if c is a constant, then the following functions are **also continuous** at a:

  1. $f+g$; 2. $f-g$; 3. $c\,f$; 4. fg; 5. f/g if $g(a)\neq 0$.

- Moreover, if g is continuous at a and if f is continuous at $g(a)$ then the **composite function** $f\circ g$ is continuous at a.

- This is a consequence of the corresponding **Limit Laws**:

$$\lim_{x\to a}\big((f+g)(x)\big) = \lim_{x\to a}\big(f(x)+g(x)\big) \overset{(1.)}{=} \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
$$= f(a)+g(a) = (f+g)(a).$$

- $\lim_{x\to 2} x\sqrt{20-x^2} = 2\cdot\sqrt{16} = 8$:
- The function $f(x) = x\sqrt{20-x^2}$ is continuous at 2 since x and x^2 are continuous at 2, and $\sqrt{x}$ is continuous at 16.

Continuity on an Interval.

- One-sided limits allow for a one-sided notion of continuity.

- A function $f\colon D \to \mathbb{R}$ is **continuous from the left** at a point $a \in D$ if

$$\lim_{x \to p^-} f(x) = f(p) \quad \text{(or } \lim_{x \to p^+} f(x) = f(p)\text{)}.$$

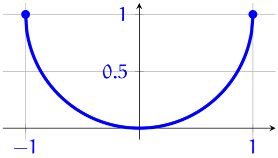
- One-sided continuity suffices for an endpoint of an interval.

- A function f is **continuous on an interval** I if it is continuous at every point a in I.

- Geometrically, you can think of a function that is continuous on an interval as a function whose graph can be drawn without lifting the pen from the paper.

Example.

- **Show** that $f(x) = 1 - \sqrt{1-x^2}$ is continuous on $[-1, 1]$.



- **Solution:** For $-1 < a < 1$, using the **Limit Laws**, we get

$$\lim_{x\to a} f(x) = \lim_{x\to a} (1 - \sqrt{1-x^2}) = \cdots = 1 - \sqrt{1-a^2} = f(a).$$

- **One-Sided Limit Laws** yield

$$\lim_{x\to -1^+} f(x) = 1 = f(-1) \quad \text{and} \quad \lim_{x\to 1^-} f(x) = 1 = f(1).$$

- Therefore, f is **continuous** on $[-1, 1]$.

Most Functions are Continuous.

- Most of our functions are continuous most of the time.

- **polynomials** and **rational functions**,
- **power** and **root functions**,
- **trigonometric functions** and their **inverses**,
- **exponential** and **logarithmic functions**
- are **continuous at every point in their domain**.

- **Example.** Where is $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?
- **Solution.** $\ln x$ is continuous for $x > 0$ and $\tan^{-1} x$ on all of $\mathbb{R}$. Hence the sum $\ln x + \tan^{-1} x$ is continuous on $(0, \infty)$.

  The denominator $x^2 - 1$ is polynomial, hence continuous on $\mathbb{R}$. Thus f is continuous on $\{x > 0 \mid x \neq \pm 1\} = (0, 1) \cup (1, \infty)$.

Making a Function Continuous.

- **Example.** For which values of c, if any, is the function

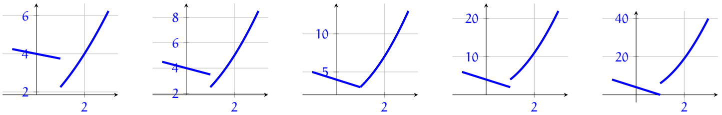
$$f(x) = \begin{cases} 4 - cx, & \text{if } x < 1, \\ cx^2 + x + 1, & \text{if } x \geqslant 1. \end{cases}$$

continuous on all of $\mathbb{R}$?

- **Solution.** f is continuous on $(-\infty, 1)$ as $4 - cx$ is continuous for every choice of c.
- Similarly, f is continuous on $(1, \infty)$ as $cx^2 + x + 1$ is continuous.
- At $a = 1$, we have $\lim_{x \to 1^-} = 4 - c \cdot 1 = 4 - c$ and

$$\lim_{x \to 1^+} = f(1) = c \cdot 1^2 + 1 + 1 = c + 2.$$

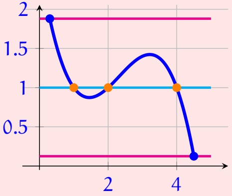
- This f is continuous at $a = 1$ only if $4 - c = c + 2$, i.e., $\underline{c = 1}$.



Intermediate Values

- **Intermediate Value Theorem:** Suppose that f is continuous on the closed interval $[a, b]$ and that $f(a) < N < f(b)$.

  Then $f(c) = N$ for some number c in the open interval (a, b).



- The version where $f(a) > N > f(b)$ is also true.
- If $f(a)$ and $f(b)$ have **opposite signs** then f has a **root** in (a, b).

- **Show** that $4x^3 - 6x^2 + 3x - 2 = 0$ has a solution between 1 and 2.
- **Solution:** Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. Then

  $$f(1) = 4 - 6 + 3 - 2 = -1 < 0, \qquad f(2) = 32 - 24 + 6 - 2 = 12 > 0.$$
- Thus $f(1) < 0 < f(2)$. By the **IVT**: $f(c) = 0$ for some $c \in (1, 2)$.

Red Squirrels.

- **A Motivating Example.** A team of researchers studying **red squirrels** in Ireland has determined that the **population density** (i.e., the number of squirrels per square kilometre) in t years from now can be modelled as

$$P(t) = \frac{200t^2 + 50t + 900}{t^3 + 30}.$$

1. What is the **current** population density?
2. What will the population density be **in** 5 **years** time?
3. What can be predicted about the **long-term** population density?

- The first and the second question are relatively easy to answer: simply substitute $t = 0$, or $t = 5$ in $P(t)$.
- In order to answer the third question, we need to apply the concept of a **limit at infinity**.

As x Gets Very Large.

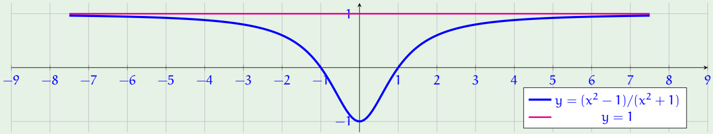
- The idea of a limit can be used to study $f(x)$ as x gets very large.

- **Example.** Consider the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

- Some values are in the table.
- It looks like $f(x)$ tends to 1 as x goes to ∞, or to $-\infty$.

x	$f(x)$
0	-1
± 1	0
± 2	0.6
± 3	0.8
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998



- **Symbolically**, $\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$. And also $\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$.

Limits at Infinity.

- Here is the **intuitive definition of a limit at infinity**:

- Suppose that L is a number and that f is a function defined on some interval (a, ∞).
- We say that **the limit of $f(x)$ as x approaches ∞ is L**, and write

$$\lim_{x \to \infty} f(x) = L$$

if we can make $f(x)$ **as close to L as we like**, by taking x to be **sufficiently large**.

- Roughly speaking, $f(x)$ approaches L as x approaches ∞:

$$f(x) \to L \text{ as } x \to \infty.$$

- Similarly, $\lim_{x \to -\infty} f(x) = L$ means $f(x) \to L$ as $x \to -\infty$.

- The line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

What Do You See?

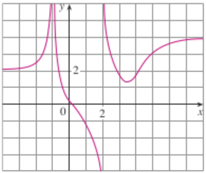


FIGURE 5

- **Example.** Describe the **infinite limits**, **limits at infinity** and **asymptotes** for the function f whose graph is shown in Figure 5.

- **Solution.** It looks like $f(x)$ becomes large as $x \to -1$, from both sides, so $\lim_{x\to -1} f(x) = \infty$.
- Also, $f(x) \to -\infty$ as x approaches 2 from the left, but $f(x) \to \infty$ as $x \to 2$ from the right: $\lim_{x\to 2^-} f(x) = -\infty$ and $\lim_{x\to 2^+} f(x) = \infty$.
- As x becomes larges, it appears that $f(x) \to 4$, and as x becomes large negative, $f(x) \to 2$, so $\lim_{x\to\infty} f(x) = 4$ and $\lim_{x\to -\infty} f(x) = 2$.
- The graph has 2 **vertical asymptotes**: the lines $x = -1$ and $x = 2$, and 2 **horizontal asymptotes**: the lines $y = 2$ and $y = 4$.

Limit Laws at Infinity.

- Most of the Limit Laws are also valid when "$x \to a$" is replaced by "$x \to \infty$" or by "$x \to -\infty$".

- **Find** $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.
- **Solution.** When x is large, $1/x$ is small.

  When x is large negative, $1/x$ is small negative:

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

- If $r > 0$ is a rational number then $\lim_{x\to\infty} \frac{1}{x^r} = 0$.
- If x^r is defined for all x then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

Limits of Rational Functions.

- **Example.** Evaluate the limit of $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ as $x \to \infty$.
- **Solution.** The **Limit Laws** can be applied after both numerator and denominator have been divided by x^2:
- $$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{(3x^2 - x - 2)/x^2}{(5x^2 + 4x + 1)/x^2} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$
$$= \frac{\lim_{x\to\infty} (3 - \frac{1}{x} - \frac{2}{x^2})}{\lim_{x\to\infty} (5 + \frac{4}{x} + \frac{1}{x^2})} = \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} \frac{2}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{4}{x} + \lim_{x\to\infty} \frac{1}{x^2}} = \frac{3 - 0 - 0}{5 + 0 + 0}.$$

- **Informally,** as $x \to \infty$,
$$f(x) = \frac{(3x^2 - x - 2)/x^2}{(5x^2 + 4x + 1)/x^2} = \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \to \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}.$$

Squirrels Revisted.

- **Example.** A team of researchers studying red squirrels in Ireland has determined that the population density (i.e., the number of squirrels per square kilometre) in t years from now can be modelled as

$$P(t) = \frac{200t^2 + 50t + 900}{t^3 + 30}.$$

1. What is the current population density? $P(0) = \frac{900}{30} = 30$.
2. What will the population density be in 3 years time? $P(3) = \frac{200 \cdot 3^2 + 50 \cdot 3 + 900}{3^3 + 30} = 50$.
3. What can be predicted about the **long-term** population density? $\frac{(200t^2+50t+900)/t^3}{(t^3+30)/t^3} = \frac{200/t + 50/t^2 + 900/t^3}{1 + 30/t^3} \to \frac{0+0+0}{1+0} = 0$ as $t \to \infty$: over time, the red squirrels will disappear. ☹

Infinite Limits a Infinity

- A limit at infinity can be infinite.

- We write
$$\lim_{x \to \infty} f(x) = \infty$$
to indicate that we can make $f(x)$ **arbitrarily large** by taking x to be **sufficiently large**.
- Similarly,
$$\lim_{x \to -\infty} f(x) = \infty, \qquad \lim_{x \to \infty} f(x) = -\infty, \qquad \text{or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

- **Example.** Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.
- **Solution.** When x is large, x^3 also is large.
When x is large negative, x^3 also is large negative:
$$\lim_{x \to \infty} x^3 = \infty \text{ and } \lim_{x \to -\infty} x^3 = -\infty.$$

Some Final Examples.

- **Find** $\lim_{x\to\infty} \sin x$.
- **Solution.** As x increases, the values of $\sin x$ keep oszillating between 1 and -1.
- There is no number L such that $\sin x$ is close to L for all sufficiently large x. Neither does $\sin x$ become arbitrarily large.
- Therefore, $\lim_{x\to\infty} \sin x$ **does not exist**.

- **Find** $\lim_{x\to\infty} (x^2 - x)$.
- **Wrong:** $\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty$.
- ∞ is not a number!
- **Solution:** $\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x-1) = \infty$, as both x and $x-1$ become arbitrarily large, and so does their product, too.

Exercises.

1. For what values of k is the function $f(x)$ continuous at $x = 2$?

$$f(x) = \begin{cases} 3x^4 - 5x^3 + x + 3, & \text{if } x < 2, \\ kx^2 + 3x - 4, & \text{if } x \geqslant 2. \end{cases}$$

2. Evaluate the following limits.
   (i) $\lim_{x \to \infty} 2^x$.
   (ii) $\lim_{x \to -\infty} 2^x$.

3. Evaluate the following limits.
   (i) $\lim_{x \to \infty} \frac{2x - 9x^3}{2x + 3x^3}$.
   (ii) $\lim_{x \to \infty} \frac{x^5 - x^4 + x^3 + x^2}{x^4 + 23\,112}$.
   (iii) $\lim_{x \to \infty} \frac{12x - 18x^2 + 6}{8x + x^3 + 16}$.

4. New research on the red squirrel has called into question the population model used above. The team **now** believes the model should be

$$P(t) = \frac{200t^2 + 50t + 900}{t + 30}.$$

What is the predicted long-term population density?