# Week 5: Limits and Limit Laws

## MA161/MA1161: Semester 1 Calculus.

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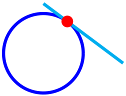
# The Tangent Problem.

- A **tangent** to a curve is a line that **touches** the curve with the **same direction** as the curve at the point of contact.
- How can this idea be made **precise**?



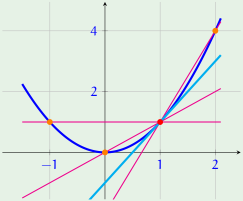
- **Find** the equation of the **tangent line** to $y = x^2$ at $P = (1, 1)$.
- **Solution.** Pick a point $Q = (x, x^2)$ nearby, $x \neq 1$.
- The **slope** of the **secant line** $PQ$ is $m = \frac{x^2-1}{x-1}$:

| $x$ | 0 | 0.5 | 0.9 | 0.99 | 0.999 | 1 | 1.001 | 1.01 | 1.1 | 1.5 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 1 | 1.5 | 1.9 | 1.99 | 1.999 | ? | 2.001 | 2.01 | 2.1 | 2.5 | 3 |

- The closer $x$ is to $1$, the closer $m$ is to $2$!
- The slope of the tangent is the **limit** of the slopes of the secants:

$$m = \lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

- Equation of the tangent line: $y = 2x - 1$.

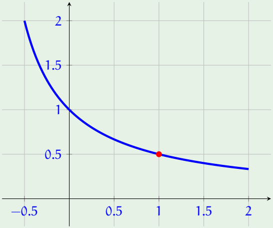


# Finding Limits.

- **Investigate**

$$f(x) = \frac{x-1}{x^2-1}$$

for $x$ near $1$, but not equal to $1$:




- | $x < 1$ | $f(x)$ | $x > 1$ | $f(x)$ |
  |---|---|---|---|
  | 0.5 | 0.666667 | 1.5 | 0.400000 |
  | 0.9 | 0.526316 | 1.1 | 0.476190 |
  | 0.99 | 0.502513 | 1.01 | 0.497512 |
  | 0.999 | 0.500250 | 1.001 | 0.499750 |
  | 0.9999 | 0.500025 | 1.0001 | 0.499975 |
  | ... | ... | ... | ... |

- The closer $x$ is to $1$ (on either side of $1$) the closer $f(x)$ is to $0.5$!
- In fact, it appears that we can make the value of $f(x)$ **as close to $0.5$ as we like** by choosing $x$ **sufficiently close to $1$**.
- **We say:** the **limit of $f(x)$ as $x$ approaches $1$** is $0.5$, and **write**

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$$

# Limits

- Here is the **intuitive definition of a limit**:

- Suppose that $L$ is a number and that the function $f(x)$ is defined for $x$ near a number $a$.
- We say **the limit of $f(x)$ as $x$ approaches $a$ is $L$**, and write

$$\lim_{x \to a} f(x) = L,$$

if we can make $f(x)$ **as close to $L$ as we like**, by taking $x$ **sufficiently close to $a$**, but **not equal to $a$**.

- Roughly speaking, $f(x)$ **approaches $L$ as $x$ approaches $a$**:

$$f(x) \to L \text{ as } x \to a.$$

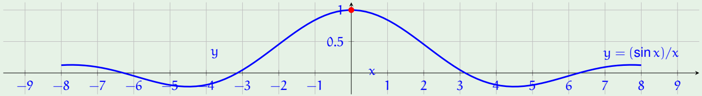
- In finding $\lim_{x \to a} f(x)$, we **never consider** $f(a)$, as $x \neq a$ by above.
- In fact, $f(x)$ need **not** be **defined** for $x = a$.
- All that matters is how $f(x)$ behaves **near** $a$, **on either side**.
- Not all limits do exist . . .

# The Trigonometric Limit

- Sometimes, $f(a)$ is not even defined, as $a$ is not in the domain of $f$. Still, the limit of $f(x)$ as $x \to a$ might exist.

- **Guess** the value of $\lim_{x \to 0} \frac{\sin x}{x}$.
- It doesn't matter that $f(0)$ is not defined.
- We can calculate $f(x)$ for many $x$ close to $0$.
- Watch out for rounding errors!
- Plotting the points obtained in this way, yields points on the graph of $f(x)$:

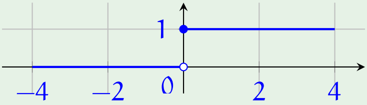
| $x$ | $(\sin x)/x$ |
|---|---|
| $\pm 1.0$ | $0.84147098$ |
| $\pm 0.5$ | $0.95885107$ |
| $\pm 0.4$ | $0.97354585$ |
| $\pm 0.3$ | $0.98506735$ |
| $\pm 0.2$ | $0.99334665$ |
| $\pm 0.1$ | $0.99833416$ |
| $\pm 0.05$ | $0.99958338$ |
| $\pm 0.01$ | $0.99998333$ |
| $\pm 0.005$ | $0.99999583$ |
| $\pm 0.001$ | $0.99999983$ |




- This suggests that $\lim_{x \to 0} f(x) = 1$.

# The Heaviside Function.

- **Recall:** $H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geqslant 0. \end{cases}$




- $H(0)$ **is defined** and equals $1$.
- However, $1$ **is not the limit** of $H(t)$ as $t \to 0$: any choice of $t < 0$, as close as you like to $0$, gives $H(t) = 0$, far from $1$.
- For a similar reason, $0$ **is not the limit** of $H(t)$ as $t \to 0$ either.
- In fact, the limit of $H(t)$ as $t \to 0$ **does not exist**.

- But: $H(t)$ approaches $1$ as $t$ approaches $0$ **from the right**.
- And $H(t)$ approaches $0$ as $t$ approaches $0$ **from the left**.
- These are **one-sided limits**: $\lim_{t \to 0^+} H(t) = 1$ and $\lim_{t \to 0^-} H(t) = 0$.
- Note how the **tiny superscripts** of $0$ make a difference!

# Left-Hand Limits and Right-Hand Limits.

- We say the **left-hand limit of $f(x)$ as $x$ approaches $a$** (or the **limit of $f(x)$ as $x$ approaches $a$ from the left**) is $L$, and write

$$\lim_{x \to a^-} f(x) = L,$$

  if we can make $f(x)$ **as close to $L$ as we like**, by taking $x$ **sufficiently close to $a$**, with $x$ **less than $a$**.
- We say the **right-hand limit of $f(x)$ as $x \to a$** $(\ldots)$ is $L$, and write

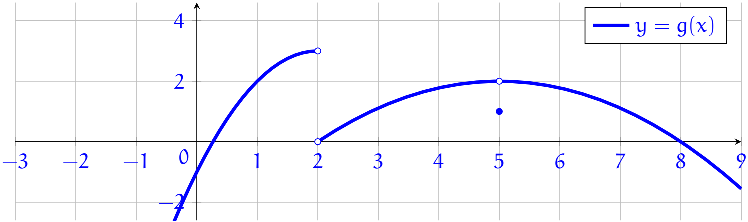
$$\lim_{x \to a^+} f(x) = L,$$

  if . . . , with $x$ **bigger than $a$**.

- Comparing the definitions, the following is obviously true:

- $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ **and** $\lim_{x \to a^+} f(x) = L$.

# Two-Sided Limits.




- The graph of a function $g$ is shown above. **Find** (if they exist):

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

- **Solution:** From the graph, $\lim_{x\to 2^-} g(x) = 3$ and $\lim_{x\to 2^+} g(x) = 0$.
- As $3 \neq 0$, the two-sided limit $\lim_{x\to 2} g(x)$ **does not exist**.
- Also, $\lim_{x\to 5^-} g(x) = \lim_{x\to 5^+} g(x) = 2$ whence $\lim_{x\to 5} g(x) = 2$, too.

# Limit Laws.

- Here is a set of nice laws that allow us to **calculate** many limits.

- Suppose that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

1. $\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. **Sum Law**
2. $\lim_{x\to a}\big(f(x)-g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$. **Difference Law**
3. $\lim_{x\to a}\big(c\,f(x)\big) = c\lim_{x\to a} f(x)$ for a constant $c$. **Constant Multiple L.**
4. $\lim_{x\to a}\big(f(x)\,g(x)\big) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$. **Product Law**
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$. **Quotient Law**

- **Sum Law:** The **limit of a sum** is the **sum of the limits**.

  If $f(x)$ is close to $L$ and $g(x)$ is close to $M$,

  then $f(x)+g(x)$ is close to $L+M$ . . .

# Calculating Limits.

6. $\lim_{x\to a} c = c$ for any constant $c$;  and 7. $\lim_{x\to a} x = a$  are obvious.

- The **Power Law** is a consequence of the Product Law, for $n \in \mathbb{N}$:

8. $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$;  in particular, 9. $\lim_{x\to a} x^n = a^n$.

10. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, for $n \in \mathbb{N}$.  **Root Law**

- **Find** $\lim_{x\to 5}(2x^2 - 3x + 4)$, justifying each step.
- **Solution:** $f(x) = 2x^2 - 3x + 4$ is a polynomial.
- $\lim_{x\to 5} f(x) = \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4$  (by 2. and 1.)

  $= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + 4$  (by 3. and 6.)

  $= 2\cdot 5^2 - 3\cdot 5 + 4 = \underline{39}$  (by 9. and 7.)

# Direct Substitution.

- With $f(x) = 2x^2 - 3x + 4$, we have $f(5) = 39 = \lim_{x \to 5} f(x)$.
- We could have gotten the limit by **directly substituting** $5$ for $x$.

- **Direct Substitution Property:**

  If $f$ is a **polynomial** and if $a$ is a point in the domain of $f$, then
  $$\lim_{x \to a} f(x) = f(a).$$

- Many other types of functions have the Direct Substitution Property, e.g., **rational functions** (see later: continuity).

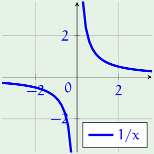
- $$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x} = \cdots = -\frac{1}{11}$$

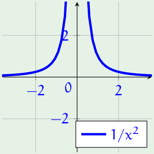
# Infinite Limits.

- If a function $f(x)$ is defined on both sides of a number $a$ then
$$\lim_{x \to a} f(x) = \infty$$
means that we can make $f(x)$ **arbitrarily large** by taking $x$ **sufficiently close** to $a$, but **not equal to** $a$.
- Similarly, $\lim_{x \to a} f(x) = -\infty$ means that we can make $f(x)$ **arbitrarily large negative** by taking $x$ sufficiently close to $a$ ...

- There are also one-sided infinite limits $\lim_{x \to a^{\pm}} = \pm\infty$ ...

- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$.
- $\lim_{x \to 0} \frac{1}{x^2} = \infty$.



# Vertical Asymptotes.

- The (vertical) line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if **at least one of the four** statements

$$\text{(i)} \lim_{x\to a^+} f(x) = \infty, \qquad \text{(ii)} \lim_{x\to a^-} f(x) = \infty,$$
$$\text{(iii)} \lim_{x\to a^+} f(x) = -\infty, \qquad \text{(iv)} \lim_{x\to a^-} f(x) = -\infty$$

is true.

- The $y$-axis $x = 0$ is a vertical asymptote of $y = \frac{1}{x^2}$ by (i), (ii).
- $x = 0$ is a vertical asymptote of $y = \frac{1}{x}$ by (i), (iv).
- $x = (n + \frac{1}{2})\pi$, for $n \in \mathbb{Z}$, is an asymptote of $y = \tan x$ by (ii), (iii).
- $x = 0$ is a vertical asymptote of $y = \ln x$ by (iii).
- $y = \frac{2x^4 - x^2 + 1}{x^2 - 4}$ has two asymptotes $x = -2$ and $x = 2$.

- Find those graphs in previous lecture notes!

# How Can a Limit Fail to Exist?

- A limit $\lim_{x\to a} f(x)$ may exist even if $f(x)$ is not defined at $x = a$.

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- Limit notation can be used to indicate **why** a limit does not exist:
- $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x) \implies \lim_{x\to a} f(x)$ does not exist.
- $\lim_{x\to a} f(x)$ does not exist if $\lim_{x\to a} f(x) = \pm\infty$.

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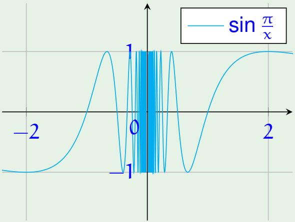
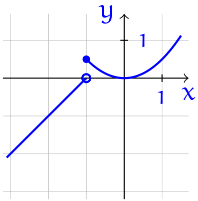
- **Example.** $\lim_{x\to 0} \sin\frac{\pi}{x}$ **does not exist**.
- $f(x) = \sin(\frac{\pi}{x})$ is not defined at $x = 0$.
- $f(10^{-n}) = \sin 10^n\pi = 0$, for $n \in \mathbb{N}$!?
- There exists **no number** $L$ such that $f(x)$ is close to $L$ **for all** $x$ close to $0$.
- Neither does $f(x)$ become arbitrarily large, nor do one-sided limits exist.



# Exercises.

1. Simplify the following expressions.
   (i) $e^{3\ln 4}$.
   (ii) $\log_4 64 + \log_2 1024$.
   (iii) $\ln(2e^{-x/2}) - \ln 2 + \frac{x}{2}$.
   (iv) $\ln 81 / \ln 3$.
2. Write down the values of
$$f(x) = \frac{\cos x}{x - \pi/2}$$
for some values of $x$ near $\pi/2$. Then guess $\lim_{x\to\pi/2} f(x)$.
3. Estimate the value of
$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}.$$
4. The graph of
$$f(x) = \begin{cases} 1+x, & x < -1, \\ \frac{1}{2}x^2, & x \geqslant -1, \end{cases}$$
is shown below. Calculate the following limits:
(i) $\lim_{x\to 0^+} f(x)$, (ii) $\lim_{x\to 0^-} f(x)$,
(iii) $\lim_{x\to 0} f(x)$, (iv) $\lim_{x\to 1^+} f(x)$,
(v) $\lim_{x\to 1^-} f(x)$, (vi) $\lim_{x\to 1} f(x)$.



# Exercises.

5. Evaluate the limit and justify each step by indicating the appropriate Limit Laws.
   (i) $\lim_{x\to 5}(4x^2 - 5x)$.
   (ii) $\lim_{x\to -3}(2x^3 + 6x^2 - 9)$.
   (iii) $\lim_{t\to 7}\dfrac{3t^2+1}{t^2-5t+2}$.
   (iv) $\lim_{u\to -2}\sqrt{9-u^3+2u^2}$.
   (v) $\lim_{x\to 3}\sqrt[3]{x+5}(2x^2-3x)$.
   (vi) $\lim_{t\to -1}\left(\dfrac{2t^5-t^4}{5t^2+4}\right)^3$.

6. Try and evaluate the following limits:
   (i) $\lim_{x\to 0}\dfrac{e^x+1}{x+1}$.
   (ii) $\lim_{x\to -1}\dfrac{x^2+x-2}{x-1}$.
   (iii) $\lim_{x\to 1}\dfrac{x^2+x-2}{x-1}$.
   (iv) $\lim_{x\to -3}\dfrac{x^2-x-12}{x+3}$.
   (v) $\lim_{x\to 5}\dfrac{x^3-3x^2-7x-15}{x-5}$.