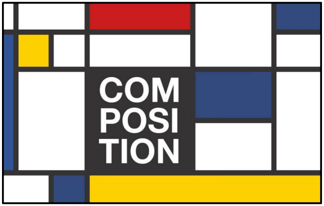
Week 4: Composition and Logarithms

MA161/MA1161: Semester 1 Calculus.

Prof. Götz Pfeiffer

School of Mathematics, Statistics and Applied Mathematics
NUI Galway

October 19–20, 2020



Function Yoga

- There are many ways to obtain **new functions from old** ones.

- We can obtain a **new function** $g(x)$ from a given function $f(x)$ by
  - **shifting**
  - **stretching**
  - **reflecting**

  the graph of $f(x)$.

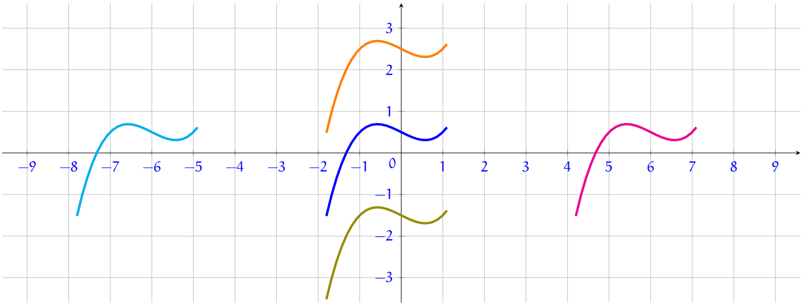
- This allows us to **quickly sketch the graphs** of many functions.

- Two functions $f(x)$ and $g(x)$ can be combined by elementary **arithmetic operations** ($+$, $-$, $\times$, $\div$) into a new function $h(x)$.
- The **composition** $f \circ g$ of functions f and g applies g first, then f to the result, i.e. f **after** g.

- We will later use these ideas to **decompose** a complex function into **simpler parts**.

Translations.

For $c > 0$, the graph of	is the **graph** of $f(x)$
$g(x) = f(x) + c$	**shifted** by c units **upward**
$g(x) = f(x) - c$	**shifted** by c units **downward**
$g(x) = f(x - c)$	**shifted** by c units to the **right**
$g(x) = f(x + c)$	**shifted** by c units to the **left**

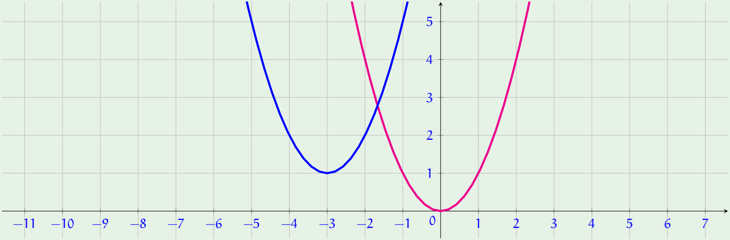


Example: Completing the Square.

- **Sketch** the graph of the function $f(x) = x^2 + 6x + 10$.
- **Solution:** By **completing the square**

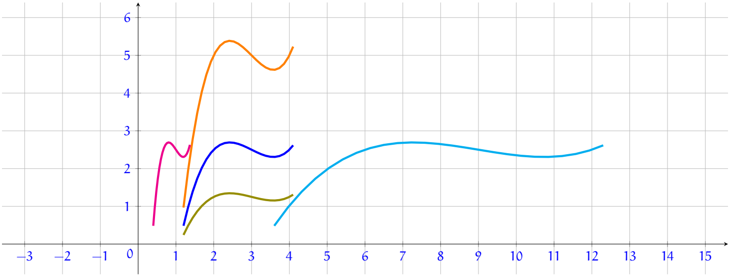
$$f(x) = (x^2 + 6x) + 10 = (x+3)^2 - 9 + 10 = (x+3)^2 + 1.$$

- Hence, if $g(x) = x^2$ then $f(x) = g(x+3) + 1$.
- The **graph** of $f(x)$ is obtained by **shifting** the **parabola** $y = x^2$ by 3 **units to the left** and by 1 **unit upwards**.

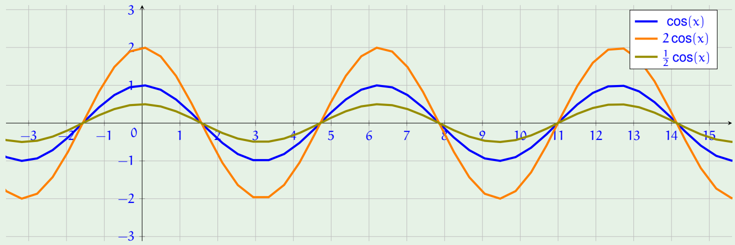


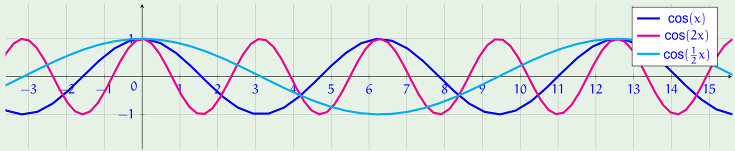
Stretches

For $c > 1$, the graph of	is the **graph** of $f(x)$
$g(x) = c\,f(x)$	**stretched** by a factor c **vertically**
$g(x) = \frac{1}{c}\,f(x)$	**shrunk** by a factor c **vertically**
$g(x) = f(cx)$	**shrunk** by a factor c **horizontally**
$g(x) = f(\frac{1}{c}x)$	**stretched** by a factor c **horizontally**



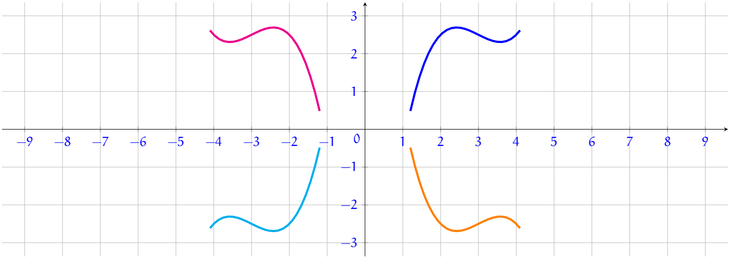
Example: Stretching $\cos(x)$. AM vs. FM





Reflections

The graph of	is the **graph** of $f(x)$
$g(x) = -f(x)$	**reflected** about the x**-axis**
$g(x) = f(-x)$	**reflected** about the y**-axis**
$g(x) = -f(-x)$	**rotated** by π about the **origin**



- **Recall:** A function f with $f(-x) = f(x)$ is an **even** function.
- A function f with $-f(-x) = f(x)$ is an **odd** function.

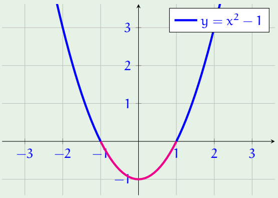
Example: Absolute Value

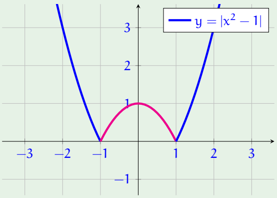
- The **absolute value** of a function $f(x)$ is the function

$$g(x) = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

- To obtain the graph of $|f(x)|$, **reflect** the part of the graph of $f(x)$ that lies below the x-axis about the x-axis.

- **Sketch** the graph of the function $g(x) = |x^2 - 1|$.





Function Arithmetic

- The **sum** $f+g$, **difference** $f-g$, **product** $f \cdot g$ and **quotient** f/g of functions f and g are defined by

$$(f+g)(x) = f(x)+g(x), \qquad (f-g)(x) = f(x)-g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x), \qquad (f/g)(x) = f(x)/g(x),$$

  for any x in the domain.

- Suppose that f has domain $A \subseteq \mathbb{R}$ and that g has domain $B \subseteq \mathbb{R}$.
- The domain of $f+g$, $f-g$ and $f \cdot g$ is the **intersection** $A \cap B$.
- The domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

- If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$ then $(f+g)(x) = \sqrt{x} + \sqrt{2-x}$ has domain $[0,\infty) \cap (-\infty, 2] = [0,2]$.
- $(f/g)(x) = \frac{\sqrt{x}}{\sqrt{2-x}}$ has domain $\{x \in [0,2] \mid x \neq 2\} = [0,2)$.

Composition of Functions.

- Suppose that $y = f(u) = \sqrt{u}$ and that $u = g(x) = x^2 + 1$.
- Then y is a function of u which is a function of x.
- Ultimately, y is a function of x:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}.$$

- This procedure is called **composition** of functions.

- The **composite** $f \circ g$ of two functions f and g is defined by

$$(f \circ g)(x) = f(g(x)).$$

- We say "f after g" for $f \circ g$, and pronounce $f(g(x))$ as "f of g of x".
- It means that we first apply g to x and then apply f to the result: f **after** g, for short.

Examples of Composites.

- **Example.** If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.
- **Solution:** $(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$.
  $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$.

- Note that, in general, $f \circ g \neq g \circ f$.
- It is possible to **compose three or more** functions, e. g.,

$$(f \circ g \circ h)(x) = f(g(h(x))).$$

- **Example.** Given $F(x) = \cos^2(x + 9)$, find functions f, g and h such that $F = f \circ g \circ h$.
- **Solution:** Let $f(x) = x^2$, $g(x) = \cos x$, $h(x) = x + 9$.
- Then $f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) = \cos^2(x + 9)$.

One-to-One Functions.

- **Example.** Size N of a bacteria culture over time (t in hours):

t	0	1	2	3	4	5	6	7	8
$N(t)$	100	168	259	358	445	509	550	573	586

- Time t required for the population N to reach a certain size:

N	100	168	259	358	445	509	550	573	586
$t(N)$	0	1	2	3	4	5	6	7	8

- A change of viewpoint is possible because at different points in time the populations differ in size: $N = f(t)$ becomes $t = f^{-1}(N)$.

- A function $f\colon D \to \mathbb{R}$ is called a **one-to-one function** if it never takes on the same value twice, that is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2 \text{ in } D.$$

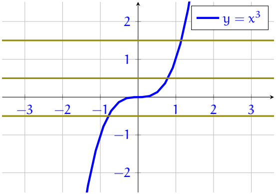
- A function is one-to-one if it passes the **horizontal line test.**

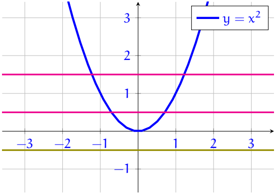
Horizontal Line Test.

- **Example.** Is the function $f(x) = x^3$ one-to-one?
- **Solution:** Two different numbers cannot have the same cube: if $x_1 \neq x_2$ then $x_1^3 \neq x_2^3$. Therefore, $f(x) = x^3$ **is one-to-one**.

- **Example.** Is the function $g(x) = x^2$ one-to-one?
- **Solution:** For instance, $g(1) = 1 = g(-1)$. Therefore, $g(x) = x^2$ **is not one-to-one**.

- **Horizontal Line Test:**





Inverse Functions.

- Suppose f is a **one-to-one** function with **domain** A and **range** B. Then its **inverse function** f^{-1} has **domain** B and **range** A and it is defined by

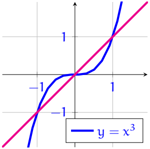
$$f^{-1}(x) = y \iff f(y) = x, \quad \text{for any } x \in B.$$

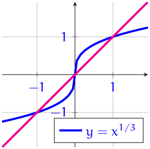
- **Do not confuse** $f^{-1}(x)$ with the **reciprocal** $f(x)^{-1} = \frac{1}{f(x)}$.
- Not every function has an inverse.

- **Find** the inverse function of $f(x) = x^3 + 2$.
- **Solution:** Write $y = x^3 + 2$.
- Solve for x: $x = \sqrt[3]{y - 2}$.
- Swap x and y: $y = \sqrt[3]{x - 2}$.
- Therefore $f^{-1}(x) = \sqrt[3]{x - 2}$.

Reflection, Inverse Yoga, Cancellation.

- The graph of $f^{-1}(x)$ is obtained by **reflecting** the graph of $f(x)$ about the line $y = x$.



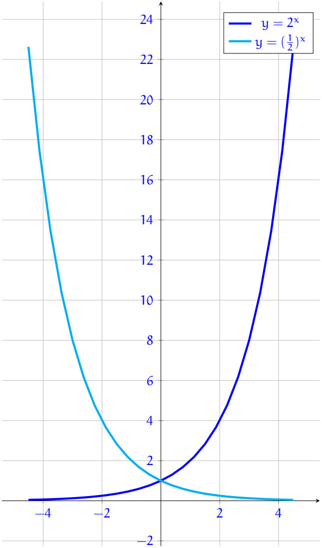


- The inverse of $f(x) = x + 5$ is $f^{-1}(x) = x - 5$.
- The inverse of $g(x) = 3x$ is $g^{-1}(x) = \frac{1}{3}x$.
- The inverse of $m(x) = -x$ is $m^{-1}(x) = -x$: thus $m^{-1} = m$.
- The inverse of $v(x) = \frac{1}{x}$ is $v^{-1}(x) = \frac{1}{x}$: thus $v^{-1} = v$.

- If a function f has domain A, range B and inverse f^{-1} then:
- $f^{-1}(f(x)) = x$ for every $x \in A$, i.e., $f^{-1} \circ f$ is the **identity** on A.
- $f(f^{-1}(x)) = x$ for every $x \in B$, i.e., $f \circ f^{-1}$ is the **identity** on B.

Exponential Functions.

- The function $f(x) = 2^x$ is called an **exponential** function, because the variable, x, is the **exponent**.
- **Not to be confused** with the power function $f(x) = x^2$, where x is in the **base**.
- One characteristic of an exponential function like $f(x) = 2^x$ is that $f(x)$ **grows very rapidly** as x increases.
- It can thus be used to model situations, where things are **doubling all the time**: $1, 2, 4, 8, 16, 32, 64, 128, \ldots$

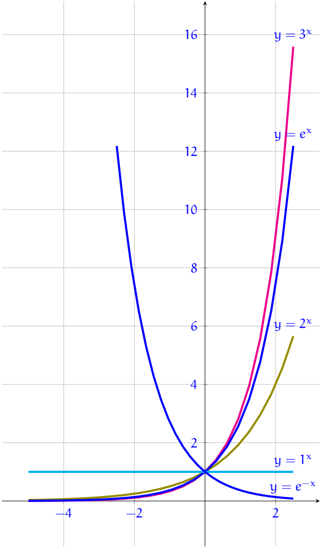


Properties of The Exponential Function $f(x) = a^x$.

- $f(0) = a^0 = 1$.
- if $x = n \in \mathbb{N}$ then $f(x) = a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}$.
- $f(1/x) = a^{1/x} = \sqrt[x]{a}$.
- If $x = p/q \in \mathbb{Q}$ then $f(x) = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.
- $f(-x) = a^{-x} = 1/a^x = (\frac{1}{a})^x$.

- Laws of Exponents:

$$a^{x+y} = a^x a^y, \qquad a^{x-y} = \frac{a^x}{a^y},$$
$$a^{xy} = (a^x)^y, \qquad (ab)^x = a^x b^x.$$



Application: Bacterial Growth.

- During the so-called **log phase** of bacterial growth in batch culture, in a controlled environment a Cyanobacteria population will double in size four times every day.
- **Show** that the population at time t, measured in hours, can be modelled as

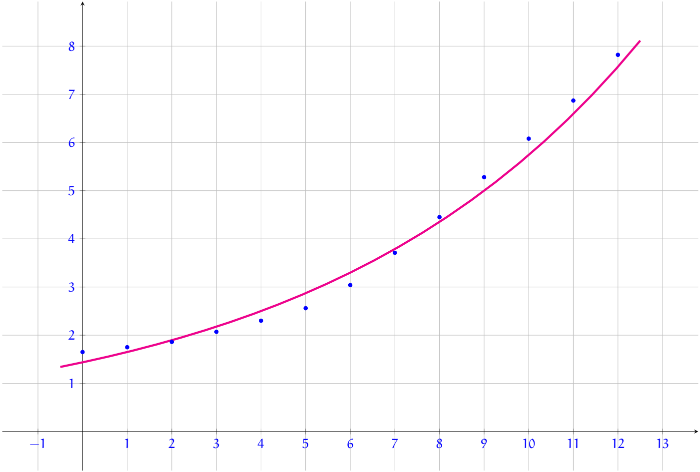
$$p(t) = 1000 \cdot 2^{t/6}.$$

if at time $t = 0$ the population is $p(0) = 1000$ cells.

- **Solution:**

$$\begin{aligned} p(0) &= 1000 = 1 \cdot 1000 = 1000 \cdot 2^{0/6} \\ p(6) &= 2\,p(0) = 2 \cdot 1000 = 1000 \cdot 2^{6/6} \\ p(12) &= 2\,p(6) = 2^2 \cdot 1000 = 1000 \cdot 2^{12/6} \\ p(18) &= 2\,p(12) = 2^3 \cdot 1000 = 1000 \cdot 2^{18/6} \\ &\vdots \\ p(t) &= 1000 \cdot 2^{t/6} \end{aligned}$$

Exponential Regression: World Population



- $P(t) = 1.43653 \cdot (1.14859)^t$ . . . doubling every 50 years . . .

Application: Radioactive Decay.

- The **half-life** of Strontium-90 is 25 years.

(a) **Find** an expression for the mass $m(t)$ that remains after t years of a sample of 24 mg of Strontium-90.

(b) **Determine** the mass remaining after 40 years.

(c) **Estimate** the time t required for the mass to be reduced to 5 mg.

- **Solution:** $m(0) = 24$.

(a) $m(25) = \frac{1}{2}m(0) = 12 = 24 \cdot 2^{-1} = 24 \cdot 2^{-25/25}$.

$m(50) = \frac{1}{2}m(25) = 6 = 24 \cdot 2^{-2} = 24 \cdot 2^{-50/25}$

$m(t) = 24 \cdot 2^{-t/25}$.

(b) $m(40) = 24 \cdot 2^{-40/25} = 7.917$ mg

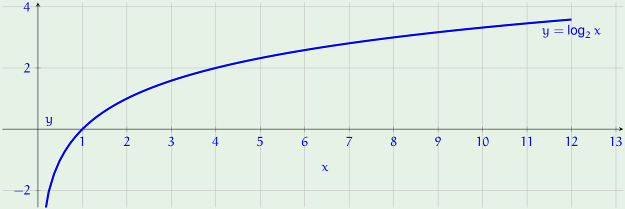
(c) $m(50) = 6$, $m(75) = 3$: between 50 and 75 years ...

Logarithmic Functions.

- Recall: The **logarithmic function** $\log_a$, with **base** $a > 1$, is the **inverse** of the **exponential function** a^x, with **base** a:

$$y = \log_a x \iff a^y = x.$$

- $\log_2 x$ is the inverse of $f(x) = 2^x$; e.g., $\log_2(8) = 3$ since $2^3 = 8$.

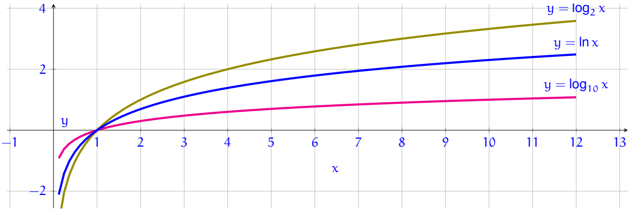


- $\log_{10}(x)$ is the **power** into which to raise 10 in order to get x:

$$\log_{10}(1\,000\,000) = \log_{10}(10^6) = 6; \qquad \log_{10}(0.001) = -3.$$

- $\log_{10}(n)$ is, roughly, the **number of digits** of $n \in \mathbb{N}$.

Properties of Logarithmic Functions.



- **Cancellation:**
  $\log_a(a^x) = x$ for all $x \in \mathbb{R}$. $a^{\log_a(x)} = x$ for $x > 0$.

- Logarithm Laws:
  $\log_a(xy) = \log_a(x) + \log_a(y)$, $\log_a(x/y) = \log_a(x) - \log_a(y)$,
  $\log_a(x^r) = r\log_a(x)$.

- $\log_2 80 - \log_2 5 = \log_2 \frac{80}{5} = \log_2 16 = 4$.

The Natural Logarithm.

- Logarithm to **base** 10 relates to **decimal representation** of numbers: $\log_{10} 10^n = n$.
- Logarithm to **base** 2 relates to **binary representation** of numbers: $64 = 2^6 = (1\,000\,000)_2 \implies \log_2 64 = 6$.
- The **natural logarithm** is $\ln x := \log_e x$, inverse to e^x.

  Here, $e = 2.71828182845905\ldots$ is **Euler's number**.

- **Change of Base Formula:** $\log_a(x) = \dfrac{\ln x}{\ln a}$.

- **Proof:** If $y = \log_a x$ then $x = a^y$ whence $\ln x = y \ln a$.

- **Find** $\log_8 5$. **Solution:** $\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$.

Applications.

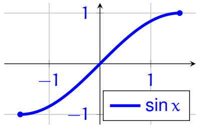
- **Solve** $e^{5-3x} = 10$.
- **Solution:** Apply $\ln$ to get $5 - 3x = \ln 10$.
  Hence $x = \frac{1}{3}(5 - \ln 10) \approx 0.8991$.

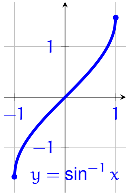
- **Bacterial Growth revisted.** A population of Cyanobacteria can double four times every day. If at time $t = 0$ the population is $p(0) = 1,000$ cells, the population after t hours is modelled as
  $$p(t) = 1000 \cdot 2^{t/6}.$$
  After how many hours does the population reach $250,000$?
- **Solution:** $250000 = 1000 \cdot 2^{t/6} \iff 2^{t/6} = 250$
  $$\iff t/6 = \log_2(250) \iff t = 6\,\log_2(250) \approx 48 \text{ hours}.$$
- **In general,**
  $$t(p) = 6\,\log_2(p/1000).$$

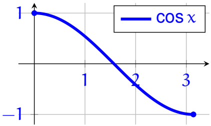
Inverse Sine and Cosine.

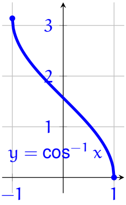
- $\sin x$ and $\cos x$ are made one-to-one by **restricting the domain**.

- The function $f(x) = \sin x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is **one-to-one**.
- Its inverse is the **inverse sine** function, denoted by $\sin^{-1}(x)$, or $\arcsin(x)$, with **domain** $[-1, 1]$ and **range** $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- The function $f(x) = \cos x$ for $x \in [0, \pi]$ is **one-to-one**.
- Its inverse is the **inverse cosine** function, denoted by $\cos^{-1}(x)$, or $\arccos(x)$, with **domain** $[-1, 1]$ and **range** $[0, \pi]$.



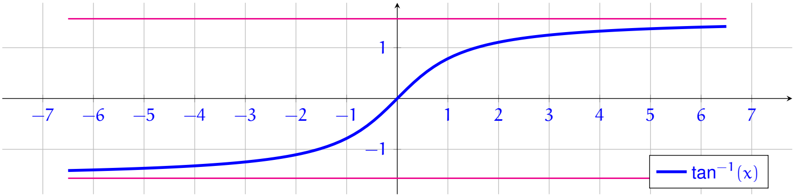






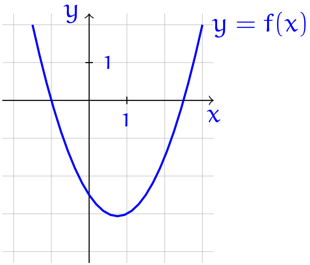
Inverse Tangent.

- The function $f(x) = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is **one-to-one**.
- Its inverse is the **inverse tangent** function, denoted by $\tan^{-1}(x)$, or $\arctan(x)$, with **domain** $\mathbb{R}$ and **range** $(-\frac{\pi}{2}, \frac{\pi}{2})$.



Exercises.

1. Here is the graph of a quadratic polynomial. What is its equation?

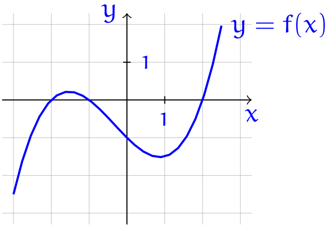


2. Find (a) $\log_{10} 1\,000\,000$, (b) $\log_2 1024$, (c) $\log_3 6561$.

3. The following graph is that of a cubic polynomial with equation

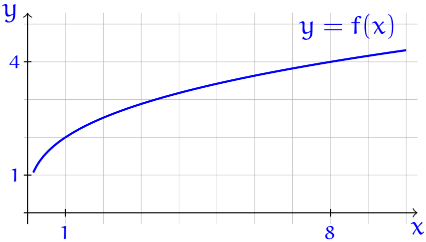
$$f(x) = K(x-a)(x-b)(x-c).$$

Find a, b, c, and K.



Exercises.

4. Here is the graph of the power function $f(x) = 2x^{1/n}$, where n is a natural number. What is the value of n?



5. (MA160/MA161 Paper 1, 2012/13) A biologist estimates that there are currently 500 zebra mussels in Lough Corrib, and that this number is doubling every year.
   (i) Show that the population can be modelled as $P(t) = 500 \times 2^t$, where t is time in years, and $t = 0$ represents the current time.
   (ii) Find P_0 and k such that the formula can be expressed as $P(t) = P_0 e^{kt}$.
6. Find, correct to 3 decimal places, (a) $\log_2 15$, (b) $\log_2 56.25$, (c) $\log_3 16$.