Week 3: A Visit to the Function Zoo.

MA161/MA1161: Semester 1 Calculus.

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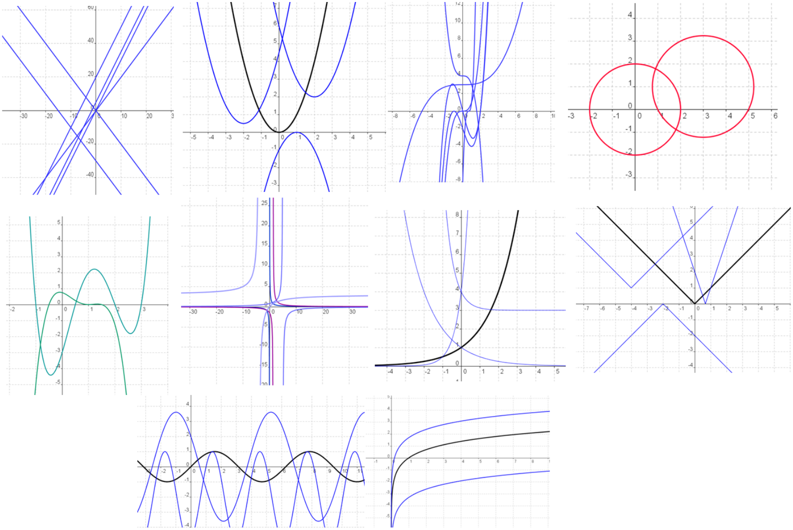
Look, it’s a Giraffe!



A Catalog of Functions.

- There are many **different types of functions** that can be used to **model relationships** between objects in the **real world**.
- We will now discuss the **behavior** and the **graphs** of the most common **essential types of functions**.
- **Examples of situations** that can be **modeled** by such functions.

1. **Linear** Functions
2. **Polynomials**
3. **Power** Functions
4. **Rational** Functions
5. **Algebraic** Functions
6. **Trigonometric** Functions
7. **Exponential** Functions
8. **Logarithms**

1. Linear Functions.

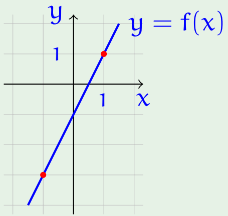
- A **linear function** is one whose graph is a straight line.
- It can be represented in **slope-intercept form** by a formula like

$$f(x) = mx + b,$$

where m is the **slope**, and b is the y**-intercept**.

- The function $f(x) = 2x - 1$ has slope 2 and y-intercept -1.

x	$f(x)$
-1.0	-3.0
-0.6	-2.2
-0.2	-1.4
0.2	-0.6
0.6	0.2
1.0	1.0



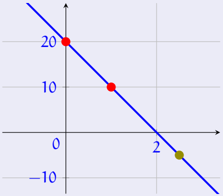
- Linear functions grow at a **constant rate**.
- A linear function with slope $m = 0$ is called a **constant** function.

Example

- As **dry air** moves upward, it **expands and cools**. If the ground temperature is $20°$C and the temperature at a height of 1 km is $10°$C, express the **temperature** T as a function of the **height** h.
- What is the temperature at a height of 2.5 km?

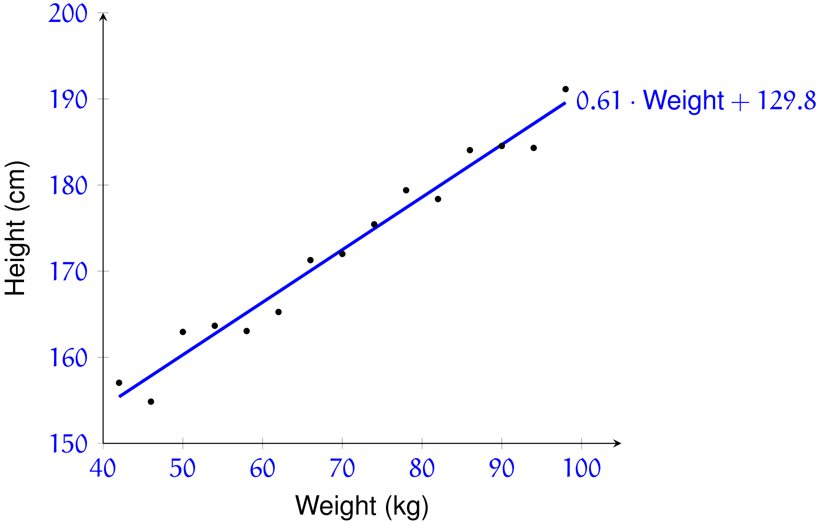
Solution.

- Assuming a **linear model**, $T(h) = mh + b$. Find b and m.
- $T = 20$ at $h = 0$ implies $20 = m \cdot 0 + b = b$. Thus $\underline{b = 20}$.
- $T = 10$ at $h = 1$ implies $10 = m \cdot 1 + 20$. Hence $\underline{m = -10}$.
- Now $T(h) = -10h + 20$ and $T(2.5) = -10 \cdot 2.5 + 20 = \underline{-5}$.



Linear Regression

- Sometimes, a line can be used to fit a given set of data points . . .



2. Polynomials.

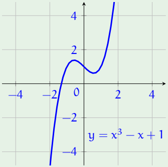
- A function P is called a **polynomial function** (or simply a **polynomial**) if it has the form

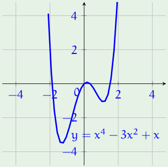
$$P(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_2\,x^2 + a_1\,x + a_0,$$

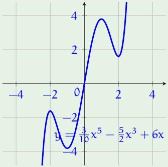
for some integer $n \geqslant 0$, and **coefficients** $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

- If the leading coefficient $a_n \neq 0$ then we say that P has **degree** n.

- The **domain** of any polynomial is $\mathbb{R}$.
- A polynomial of degree 1 is a **linear function** $P(x) = ax + b$.

- $P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 - \sqrt{2}$ is a polynomial of degree 6.
- 





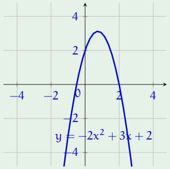
Quadratic Functions.

- Polynomials are commonly used to model various quantities in the natural and social sciences.

- The distance d travelled by a **falling object** is $d = \frac{1}{2}gt^2$.

- A polynomial of degree 2 is a **quadratic** $P(x) = ax^2 + bx + c$.
- Its graph is a **parabola**, obtained by shifting $y = ax^2$.
- The **quadratic formula** allows us to write quadratic functions in the form $P(x) = K(x - A)(x - B)$, for numbers $A, B, K \neq 0$.

- Consider $P(x) = -2x^2 + 3x + 2$.
- $P(x) = 0 \Leftrightarrow x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot (-2) \cdot 2}}{2 \cdot (-2)} = \frac{-3 \pm 5}{-4}$, i.e., $x = -\frac{1}{2}$ or $x = 2$.
- Hence $P(x) = -2(x - 2)(x + \frac{1}{2})$.



- Not every quadratic has the form $P(x) = K(x - A)(x - B)$ . . .

Cubic Functions.

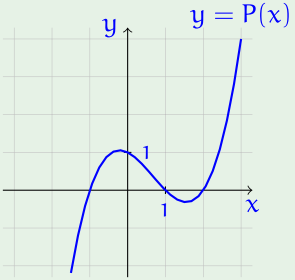
- A polynomial of degree 3 is called a **cubic function** and has the form
$$P(x) = ax^3 + bx^2 + cx + d$$
for numbers a, b, c, d, where $a \neq 0$.

- The **volume** of a sphere grows with the cube of its **radius**:
$$V = \frac{4}{3}\pi r^3.$$
- In analytical chemistry, the **Charlot equation** involves a cubic.
- **Rayleigh waves**, as produced by earthquakes, are governed by a cubic equation.

- A function of the form $P(x) = K(x - A)(x - B)(x - C)$ for any numbers $A, B, C, K \neq 0$ is a cubic function.
- Not every cubic has the form $P(x) = K(x - A)(x - B)(x - C)$ . . .

Telling a Cubic from its Graph.

- The following graph is that of a **cubic polynomial** $P(x)$.
- What is its **equation**?



- Assume $P(x)$ has an equation of the form

$$P(x) = K(x - A)(x - B)(x - C).$$

- Find A, B, C, and K!

- **Solution:** A, B, and C are the zeros of $P(x)$, hence

$$A = -1;\ B = 1;\ C = 2.$$

- $P(x) = K(x + 1)(x - 1)(x - 2)$.
- Also $P(0) = 1$ in the graph.
- On the other hand, $P(0) = K(1)(-1)(-2) = 2K$.
- It follows that $K = \frac{1}{2}$.

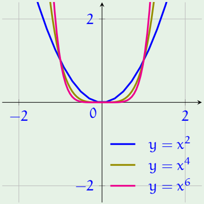
- **Answer:** $P(x) = \frac{1}{2}(x + 1)(x - 1)(x - 2) = \frac{1}{2}x^3 - x^2 - \frac{1}{2}x + 1$.

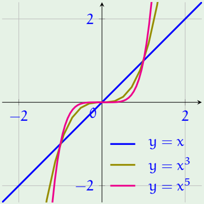
3. Power Functions.

- A **power function** is a function of the form $f(x) = x^a$ for some constant $a \in \mathbb{R}$.
- We call a the **exponent** of the function f.

- We consider some **special cases**:

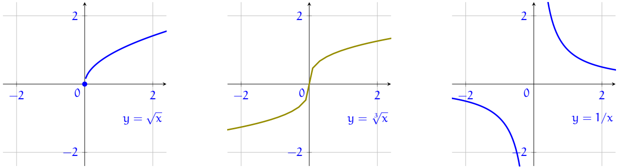
- If $a = n \in \mathbb{N}$ then f is a **polynomial**, like x^2, x^3, ...





Root Functions; the Reciprocal Function.

- If $a = 1/n$ for $n \in \mathbb{N}$ then $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**.
- The **domain** of f is $[0, \infty)$ if n is even, and $\mathbb{R}$ if n is odd.



- If $a = -1$ then $f(x) = x^{-1} = 1/x$ is the **reciprocal function**.
- The graph of f is a **hyperbola** with the coordinate axes as its **asymptotes**.

- In Mechanics, **Boyles Law** $P(V) = k/V$ states that pressure P is inversely proportional to the volume V.

4. Rational Functions.

- A **rational function** f is a quotient of polynomials P and Q,

$$f(x) = \frac{P(x)}{Q(x)}.$$

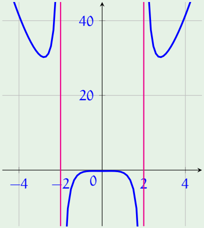
- The **domain** of the rational function $f(x) = P(x)/Q(x)$ consists of all points $x \in \mathbb{R}$ such that $Q(x) \neq 0$.

- The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain

$$\{x \in \mathbb{R} : x \neq \pm 2\} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$



5. Algebraic Functions.

- A function f is called an **algebraic function** if it can be constructed from polynomials using the **algebraic operations** of addition, subtraction, multiplication, division, and taking roots.

- **Power** functions and **rational** functions are algebraic functions.

- **Examples:** $f(x) = \sqrt{x^2+1}$; $\quad g(x) = \dfrac{x^4 - 16x^2}{x + \sqrt{x}} + (x-1)\sqrt[3]{x+1}$.
- In **relativity theory**, the **mass** m of a particle with velocity v is given by the algebraic function

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

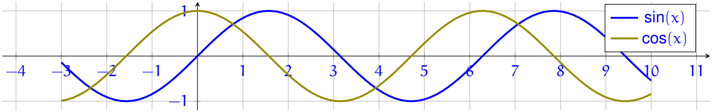
where m_0 is the mass of the particle at rest, and $c = 299\,792\,458$ m/s is the **speed of light**.

6. Trigonometric functions.

- The most important **trigonometric functions** are:

$$\sin(x); \quad \cos(x); \quad \tan(x) = \frac{\sin(x)}{\cos(x)}.$$

- In calculus, **angles** are measured in **radians**.
- The graphs of $\sin(x)$ and $\cos(x)$ look like:

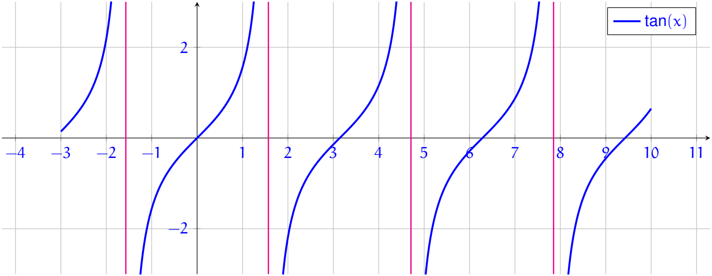


- The **domain** of both $\sin(x)$ and $\cos(x)$ is $\mathbb{R}$, the range is the closed interval $[-1, 1]$, i.e, $|\sin x| \leqslant 1$ and $|\cos x| \leqslant 1$.
- They are **periodic** with period 2π: $f(x + 2\pi) = f(x)$ when $f(x) = \sin(x)$ or $f(x) = \cos(x)$.
- This makes them suitable for modelling **repetitive phenomena**.

The Tangent Function.

- $\sin(x) = 0$ when $x = n\pi$ for $n \in \mathbb{Z}$; e.g., $x = \pi$, $x = 2\pi$, etc.
- $\cos(x) = 0$ when $x = -\pi/2$, $x = \pi/2$, $x = 3\pi/2$, ...
- $\tan(x) = \sin(x)/\cos(x)$ is undefined whenever $\cos(x) = 0$.

- The graph of $\tan(x)$ looks like:



- Note that $\tan(x)$ has **period** π: $\tan(x + \pi) = \tan(x)$.

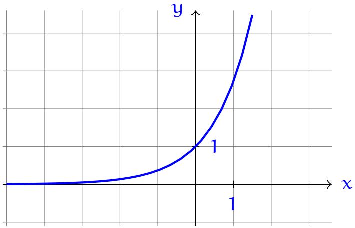
7. Exponential Functions.

- An **exponential function** is a function of the form

$$f(x) = a^x$$

for some **positive** constant **base** $a > 0$.

- The graph of $f(x) = e^x$ (where $e = 2.718281\ldots$) looks like:

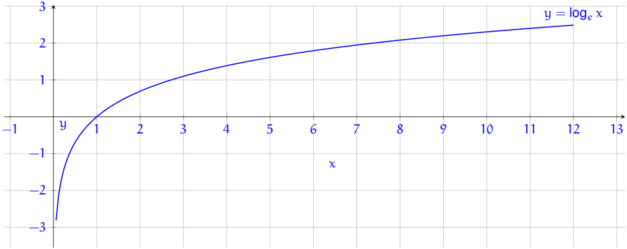


- The **domain** of e^x is all of $\mathbb{R}$, the **range** is $(0, \infty)$.
- Exponential functions are useful for modeling natural phenomena like **population growth** and **radioactive decay**, ...

8. Logarithmic Functions.

- The **logarithmic functions** $f(x) = \log_a(x)$, where the **base** a is a positive constant, are the **inverse functions** of the exponential functions: $y = \log_a(x)$ if $x = a^y$.

- The **domain** of $\log_a(x)$ is $(0, \infty)$, the range is $\mathbb{R}$.
- The graph of the **natural logarithm** $\ln(x) = \log_e(x)$ looks like:



What Type are You?

- **Classify** the following functions as one of the types of functions that we have discussed this week.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1+x}{1-\sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

Solution.

(a) $f(x) = 5^x$ is an exponential function: x is the exponent.

(b) $g(x) = x^5$ is a power function, or a polynomial of degree 5.

(c) $h(x)$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.

Exercises.

1. For each of the following functions, determine if it is even, odd, or neither.

   (i) $f(x) = \dfrac{x}{x^2+1}$,

   (ii) $f(x) = \dfrac{x^2}{x^4+1}$,

   (iii) $f(x) = x|x|$,

   (iv) $f(x) = 2 + x^2 + x^4$.

   (v) $f(x) = \dfrac{t^3+3t}{t^4-3t^2+4}$.

2. Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

   (i) $f(x) = \log_2 x$,

   (ii) $g(x) = \sqrt[4]{x}$,

   (iii) $h(x) = \dfrac{2x^3}{1-x^2}$,

   (iv) $f(x) = 1 - 1.1t + 2.54t^2$,

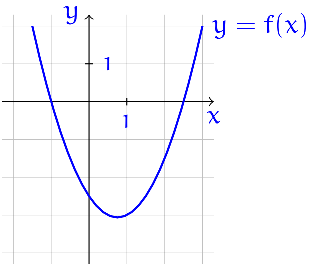
   (v) $v(t) = 5^t$,

   (vi) $w(\theta) = \sin\theta\,\cos^2\theta$.

Exercises.

3. Are the trigonometric functions sin, cos, and tan even, odd, or neither?

4. Here is the graph of a quadratic polynomial. What is its equation?



5. The following graph is that of a cubic polynomial with equation

$$f(x) = K(x - a)(x - b)(x - c).$$

Find a, b, c, and K.