Week 2: Functions.

MA161/MA1161: Semester 1 Calculus.

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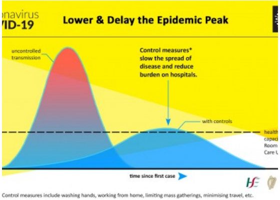
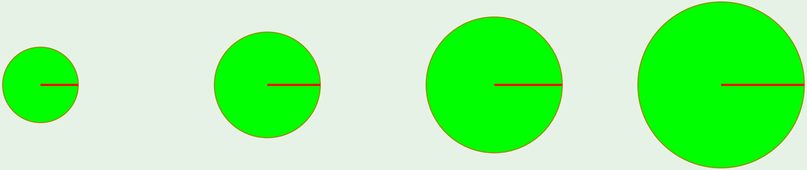


Functions.

- **Functions** arise whenever one quantity **depends** on another.



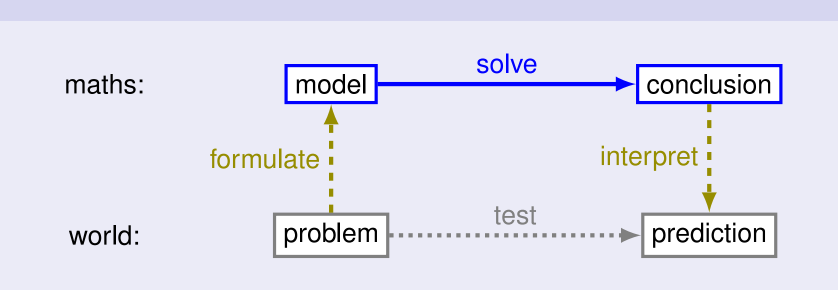
- The **area** A of a circle **depends** on the **radius** r of the circle.
- The **equation** $A = \pi r^2$ describes a **rule** that connects r and A.
- This **formula** assigns to each positive number r one value of A.
- We say that A **is a function of** r, and write

$$A(r) = \pi r^2.$$

- In this setting, A is called the **dependent variable** and r is called the **independent variable**: A depends on r.

Mathematical Models

- A **mathematical model** is a **mathematical description** (by means of a function, equation) of a **real-world phenomenon**.
- The model helps to **understand** the phenomenon, and perhaps to make **predictions** about future behavior.



- A good model **simplifies reality** to permit **mathematical calculations**, while being **sufficiently accurate** to provide **valuable conclusions**.
- Be aware of the **limitations** of the chosen model.

Function Values, Domain and Range.

- A **function** f is a **rule** that assigns to **each** element x in a set X **exactly one** element, called $f(x)$, in a set Y.
- We write

$$f\colon X \to Y$$

  and say that "f is a function from X to Y".
- The set X is called the **domain** of the function f.
- The element $f(x)$ is called the **value** of f at x (or the **image** of x under f) and is pronounced as "f of x".
- The **range** of f is the set $\{f(x) : x \in X\}$ of **all values** $f(x) \in Y$ as x varies over the domain X.

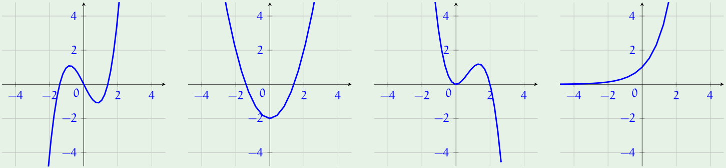
- Here, the sets X and Y are usually **sets of numbers**.
- Not every element $y \in Y$ needs to occur as a value $f(x)$.
- One element $y \in Y$ can serve as value $f(x)$ for several $x \in X$.

Examples of Functions.

1. The **age** of each student in this class is a function from the class (X is a set of students) to $Y = \mathbb{N}$, the natural numbers.
2. The **world population** $P(t)$ at any given time t, is a function from $X = \mathbb{R}$ (representing time t) to $Y = \mathbb{R}$ (measuring population).
3. **City status** legally recognised:

Dublin	Cork	Limerick	Waterford	Galway
1172	1185	1199	1202	1985

Source: Wikipedia

4. 

5. The **quadratic** function $f(x) = 2x^2 - 3x + 5$ from $\mathbb{R}$ to $\mathbb{R}$.
6. The **identity function** $f(x) = x$ has domain and range $\mathbb{R}$.

4 Ways to Represent a Function.

- A function can be represented in many differents ways:

1. **verbally** (by a description in **words**);
2. **numerically** (as a **table** of values);
3. **visually** (as a **graph**);
4. **algebraically** (by an explicit **formula**).

- Often it is possible, and useful, to go from one way to another.

Words → Formula

- A **rectangular storage container** with an open top has a volume of 10 m^3.
- Its base is twice as long as it is wide.
- **Material** for the base costs €10 per m^2, material for the side costs €6 per m^2.
- **Express** the **cost** of materials as a **function** of the **width** of the base.

Tabular Representation.

- Specific function values can be listed in a table, e.g. log table.

Table → ?

- The function g assigns to each **percentage** value an exam **grade**, according to the following table.

Percentage	Grade
70 – 100	A
60 – 69	B
55 – 59	C+
50 – 54	C−
40 – 49	D
35 – 39	E+
30 – 34	E−
0 – 29	F

- The **domain** of g is the set of percentages $\{0, 1, 2, \ldots, 100\} \subseteq \mathbb{Z}$.
- The **range** of g is the set of grades $\{\mathtt{A}, \mathtt{B}, \mathtt{C+}, \mathtt{C-}, \ldots, \mathtt{F}\}$

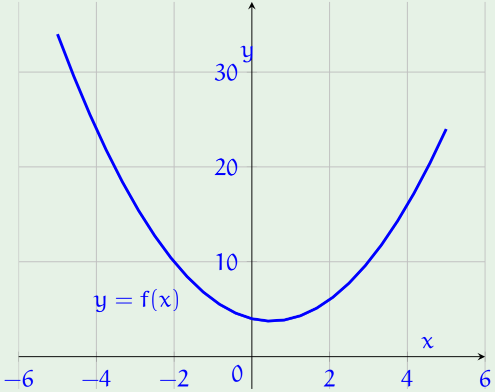
Graphical Representation.

- A common way to **visualize** a function $f\colon X \to \mathbb{R}$ is its **graph**

$$\{(x,y) : x \in X,\ y = f(x)\} \quad \subseteq \mathbb{R}^2$$

in the x, y-plane.

Graph → Table



x	$f(x)$
−4	24
−2	10
0	4
2	6
4	16

Algebraic Representation.

- Many functions are given by an algebraic **formula**.

Formula → Table

- Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by the **formula**

$$f(x) = 2x^3 - 1.$$

- Then we can **compute** and **tabulate** some values:

$$\begin{array}{ccc} f(0) = -1, & f(1) = 1, & f(10) = 1999, \\ f(-1) = -3, & f(-10) = -2001, & \dots \end{array}$$

- The **formula** of a function f can be used to **sketch its graph** . . .
- $f(x) = 2x^3 - 1$ has **domain** $\mathbb{R}$, by definition.
- The **range** of f is also all of $\mathbb{R}$, or is it?

Domain Convention.

- Often, the domain of a function is not explicitly stated.

Convention:

- The **domain** of a function f is the set of all numbers x for which $f(x)$ **makes sense** and gives a **real-number output**.

- **Find the domain** of the function
$$g(x) = \frac{1}{x^2 - x}.$$
- Since $x^2 - x = x(x-1)$, the function $g(x)$ is not defined when $x = 0$ or $x = 1$ (**division by zero!**).
- So the domain of g is
$$\{x \mid x \neq 0,\ x \neq 1\} = (-\infty, 0) \cup (0, 1) \cup (1, \infty).$$
- **Find the domain** D of $f(x) = \sqrt{x+2}$.
- $D = \{x \mid x \geqslant -2\} = [-2, \infty)$, as $\sqrt{x}$ **is not real** for negative x.

The Vertical Line Test.

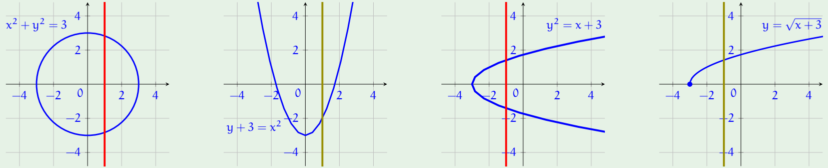
- Not every equation, table, graph defines a function.

x	2	4	5	5	6
y	3	3	2	7	6

assigns to $x = 5$ two y-values 2 and 7.

Vertical Line Test

- A curve in the x, y-plane is the graph of a function $f(x)$ if and only if any vertical line **intersects the curve at most once**.



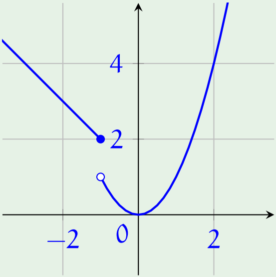
- **Recall:** A function $f\colon X \to Y$ is a rule that assigns to each $x \in X$ **exactly one** value $y \in Y$.

Piecewise Defined Functions.

- Some functions are defined by **different formulas** in different parts of their domains.
- Such functions are called **piecewise defined functions**.

- The function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$



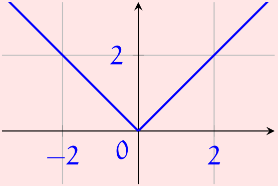
- Some values:
- $f(-2) = 1 - (-2) = 3$ since $-2 \leqslant -1$.
- $f(-1) = 1 - (-1) = 2$ since $-1 \leqslant -1$.
- $f(0) = 0^2 = 0$ since $0 > -1$.

The Absolute Value Function.

- The **absolute value** of a number a is denoted by $|a|$.
- Recall that $|a| = a$ if $a \geqslant 0$ and $|a| = -a$ if $a < 0$.
- Thus $|a|$ measures the **distance** from a to 0 on the number line.

- The **absolute value function** $f(x) = |x|$ is defined as

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{if } x < 0. \end{cases}$$



- More generally, $|x - a| = |a - x|$ is the **distance** from x to a.

- $|17 - 23| = 23 - 17 = 6$.
- $\{x \in \mathbb{R} : |x - 3| < 1\} = (2, 4)$.
- **Solve** $|x - 5| + 7 > 10$. $\{x : |x - 5| > 3\} = (-\infty, 2) \cup (8, \infty)$.

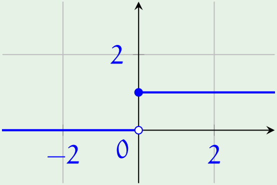
Other Piecewise Defined Functions.

- The **percentage grade** function g is piecewise defined:

$$g(x) = \begin{cases} \text{F} & \text{if } 0 \leqslant x < 30 \\ \text{E}- & \text{if } 30 \leqslant x < 35 \\ \text{E}+ & \text{if } 35 \leqslant x < 40 \\ \vdots & \vdots \\ \text{B} & \text{if } 60 \leqslant x < 70 \\ \text{A} & \text{if } 70 \leqslant x \leqslant 100 \end{cases}$$

- The **Heaviside step function** H is defined as

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geqslant 0. \end{cases}$$



Even and Odd Functions.

- A function $f\colon D \to \mathbb{R}$ is called an **even function** if it satisfies $f(-x) = f(x)$ for all x in its domain D.

- $f(x) = x^2$ is **even** because $f(-x) = (-x)^2 = x^2 = f(x)$.

- A function $f\colon D \to \mathbb{R}$ is called an **odd function** if it satisfies $f(-x) = -f(x)$ for all x in its domain D.

- $f(x) = x^3$ is **odd** because $f(-x) = (-x)^3 = -x^3 = -f(x)$.
- Most functions are **neither** even nor odd.

- Is this function **even**, **odd**, or **neither**?

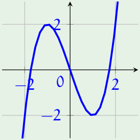
  (a) $f(x) = x^3 - 3x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Symmetries.

- We **solve the problems** from the previous slides and make an observation about the **geometry** of the graphs of these functions.

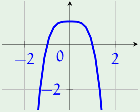
(a) $f(x) = x^3 - 3x$ is **odd**, because

$$f(-x) = (-x)^3 - 3(-x)$$
$$= -x^3 + 3x$$
$$= -(x^3 - 3x) = -f(x)$$



(b) $g(x) = 1 - x^4$ is **even**, because

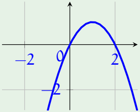
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$



(c) $h(x) = 2x - x^2$ is **neither**, because

$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

is different from both $h(x)$ and $-h(x)$.



- The graph of an **even** function is **symmetric** about the y**-axis**.
- That of an **odd** function is **symmetric** about the **origin** $(0, 0)$.

Increasing and Decreasing Functions.

- A function f is called **increasing** on an **interval** I, if

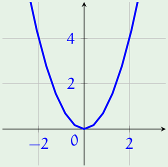
$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

- A function f is called **decreasing** on I, if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

- In these definitions, the inequalities between the function values $f(x_1)$ and $f(x_2)$ must be satisfied for **every pair** of numbers x_1 and x_2 in I with $x_1 < x_2$.

- The function $f(x) = x^2$ is **decreasing** on the interval $(-\infty, 0]$, and **increasing** on the interval $[0, \infty)$.



Exercises.

1. For each of the following functions, determine the largest possible domain and range as subsets of $\mathbb{R}$.
   (i) $f(t) = 1/(1+t)$,
   (ii) $f(x) = \sqrt{9-x^2}$,
   (iii) $f(x) = \cos(x)$,
   (iv) $f(t) = \sin(5t-2)$,
   (v) $f(x) = 1 + (1-x^2)^{-1}$,
   (vi) $f(x) = e^x$.
2. Find the domain of the following functions:
   (i) $f(x) = \dfrac{x}{3x-1}$,
   (ii) $f(x) = \dfrac{1+x}{x^2-3x+2}$,
   (iii) $f(x) = \sqrt{4-x^2}$,
   (iv) $f(x) = \sqrt{x} + \sqrt{4-x}$.
3. Find the largest possible domain and range for the following functions and sketch them:
   (i) $f(x) = x\,|x|$,
   (ii) $f(x) = \dfrac{1}{|x|+1}$,
   (iii) $f(x) = \dfrac{1}{|x+1|}$,
   (iv) $f(x) = \begin{cases} x+1, & \text{if } x < 0, \\ 1-x, & \text{if } x \geqslant 0, \end{cases}$
   (v) $f(x) = \begin{cases} x+9, & \text{if } x < -3, \\ -2x, & \text{if } |x| \leqslant 3, \\ -6, & \text{if } x > 3. \end{cases}$

Exercises.

4. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe (in words) how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.
5. (MA101 Summer Exam 2011/2012) The four graphs below are of the following functions: $y = 2^x$, $y = x^{-2}$, $y = \sqrt{x}$ and $y = x^2$, but not necessarily in that order. Which is which?