Week 1: Introduction; Numbers, Inequalities.

MA161/MA1161: Semester 1 Calculus.

Prof. Götz Pfeiffer

School of Mathematics, Statistics and Applied Mathematics
NUI Galway

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Welcome to MA161 - Mathematical Studies.

- **MA161** is **Mathematical Studies**: a mathematics module for students in 1st year **Science**.
- (**MA1161** is **Mathematical Studies** for 1st year **Project and Construction Management**.)
- MA161 has four parts:

1. **Semester 1 Calculus**: Mon and Tue.
2. **Semester 1 Algebra**: Wed and Thu.
3. **Semester 2 Calculus.**
4. **Semester 2 Algebra/Probability/Statistics.**

- Your Lecturer for Semester 1 Calculus:
  Prof. Götz Pfeiffer
  School of Mathematics, Statistics and Applied Mathematics.
  Room ADB-1008, Arás de Brún
  Email: `goetz.pfeiffer@nuigalway.ie`

What, Where and When.

Lectures: All lectures are **online**.
Calculus: Mon 1pm, Tue 10am.
Algebra: Wed 10am, Thu 10am.

Tutorials: **to be arranged . . .**

Online resources for MA161 **Semester 1 Calculus** can be found under the “Calculus 1” links on this course’s **blackboard** page:

- these **slides**,
- the lecture **videos**,
- **problem sheets** (discussed in the tutorials),
- links to **assignments** (online, to be submitted by Friday),
- further **announcements**,
- etc.

What is Calculus?

Calculus is the mathematical study of **continuous change**.

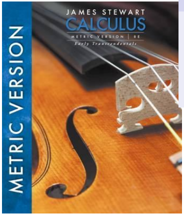
- Calculus has two major branches:
- **Differential Calculus**: **rates of change** and **slopes of curves**.
- **Integral Calculus**: **accumulation** of quantities and the **areas under** and **between curves**.
- These two branches are related to each other by the **Fundamental Theorem of Calculus**.
- A fundamental notion in both branches is that of **convergence** of certain **infinite sequences** and **series** to a well-defined **limit**.
- Today, calculus has widespread uses in **science**, **engineering** and **economics** and can solve many problems that algebra alone cannot.
- Therefore, calculus forms an important part of **modern mathematics education**.

Topics, Textbook.

- The **key topics** in MA161 Semester 1 Calculus are as follows.

1. Real Numbers and Functions,
2. Limits and Derivatives,
3. Differentiation Techniques,
4. Applications of Differentiation.

- These correspond to chapters 1–4 of the **textbook**.



- James Stewart
- **Calculus: Early Transcendentals 8th Ed.**

- There are copies in the library.
- Can be bought in bookshops or online.
- Download chapters from the library website.

Numbers

- Calculus is based on the **real number system**.

- The **natural numbers**

$$1, 2, 3, 4, 5, \dots$$

are used for **counting** and denoted by $\mathbb{N}$.

- The set of all **integers**

$$\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots$$

is denoted by $\mathbb{Z}$.

- The **rational numbers**, denoted by $\mathbb{Q}$, are constructed as **quotients** of integers: a rational number r can be expressed as

$$r = \frac{m}{n}$$

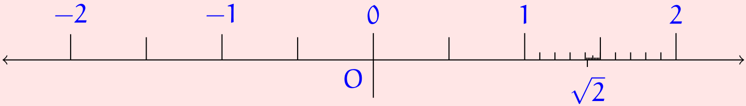
where m is an **integer** and n is a **natural number**.

- **Examples:** $\frac{1}{3} = 0.333\dots, \quad 0.17 = \frac{17}{100}, \quad -\frac{3}{7} = \frac{-3}{7}, \quad 19 = \frac{19}{1}.$

Real Numbers

- When we say "number" in this course, we mean "real number".

- The set of all **real numbers** is denoted by $\mathbb{R}$.
- A real number can be seen as a **point** on the **number line**:



- Here, the number 0 corresponds to a chosen point O, the **origin**.
- A **positive** number x is represented by the point at **distance** x to the **right** of O.
- A **negative** number $-x$ is the point at distance x to the **left** of O.

- Some numbers, such as $\sqrt{2}$ and π, cannot be expressed as quotients of integers, these are called **irrational numbers**:

$$\sqrt{2} = 1.41421356\ldots$$

A Short Review of Sets

- Calculus makes frequent use of **set notation**.

- A **set**, roughly speaking, is a **collection of objects**.
- These objects are called the **elements** of the set.

- Some sets can be described by **listing all their elements**.

- For instance, the set A consisting of **all positive integers less than** 7 can be written as
$$A = \{1, 2, 3, 4, 5, 6\}.$$
- Alternatively, we could write A in **set-builder notation** as
$$A = \{x \in \mathbb{Z} \mid 0 < x < 7\},$$
which reads "A is the set of all integers x such that $0 < x < 7$".

- The **empty set**, denoted by $\varnothing$, contains no element at all: $\varnothing = \{\}$.

Elements, Union, Intersection.

- If A is a set, the notation

$$x \in A$$

means that x **is an element of** A, and

$$x \notin A$$

means that x **is not an element of** A.

- If $A = \{1, 2, 3, 4, 5, 6\}$ then $3 \in A$ but $7 \notin A$.

- If A and B are sets, their **union** $A \cup B$ is the set of all elements that are in A or in B (or in both): $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- The **intersection** of A and B is the set $A \cap B$ of elements contained in both A and B: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Subsets.

- A set T is a **subset** of the set S if every element of T is an element of S. We then write $T \subseteq S$.

- The set $T = \{2, 8\}$ is a subset of the set $S = \{2, 4, 8\}$.

Let $S = \{-1, 2, 7, 13\}$. Which T_i is a subset of S?

- $T_1 = \{-1, 7, 13\}$? ✔ All of $-1, 7, 13$ are in S.
- $T_2 = \{1, 2, 13\}$? ✗ -1 is in S but 1 is not.
- $T_3 = \{13, 7, 2, -1\}$? ✔ The order does't matter.
- $T_4 = \{-1, 2, 7, 13, 15\}$? ✗ $S \subseteq T_4$ but $T_4 \not\subseteq S$.
- $T_5 = \{2\}$? ✔ T_5's only element is in S.
- $T_6 = 7$? ✗ $T_6 \in S$ but $T_6 \not\subseteq S$.

Intervals.

- In Calculus, an **interval** is a popular type of sets of numbers.
- On the number line, intervals correspond to **line segments**.
- Here it matters, whether the **end points** are **included** or not.

- If $a < b$, the **open interval** from a to b consists of all numbers (strictly) between a and b and is denoted by

$$(a, b) = \{x \mid a < x < b\}.$$

- The **closed interval** from a to b is the set

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}.$$

- Round **parentheses** $(\ )$ indicate: the end points are **excluded**.
- Square **brackets** $[\ ]$ indicate: the end points are **included**.
- The notation can be used to **include only one** end point.
- We also consider **infinite intervals** like $(a, \infty) = \{x \mid x > a\}$.
- **However**, this does not mean that infinity (∞) is a number.

A Table of Interval Types

$(a, b) = \{x \mid a < x < b\}$	
$[a, b] = \{x \mid a \leqslant x \leqslant b\}$	
$(a, b] = \{x \mid a < x \leqslant b\}$	
$[a, b) = \{x \mid a \leqslant x < b\}$	
$(a, \infty) = \{x \mid x > a\}$	
$[a, \infty) = \{x \mid x \geqslant a\}$	
$(-\infty, b) = \{x \mid x < b\}$	
$(-\infty, b] = \{x \mid x \leqslant b\}$	
$(-\infty, \infty) = \mathbb{R}$	

Solving Inequalities

- The real numbers are **ordered**.

- We say a is **less than** b and write $a < b$ if $b - a$ is positive,

- This means that a lies to the left of b on the number line.

- $a > b$ means the same as $b < a$.
- $a \leqslant b$ means a is less than or equal to b.
- $a \geqslant b$ means the same as $b \leqslant a$.

- An **inequality** is like an equation, except that one of $<, >, \leqslant, \geqslant$ replaces the equality sign, e.g.: $3x + 4 < 7$.
- To **solve** an inequality that involves an unknown x means to **determine all** numbers x for which the **inequality is true**.

Rules for Inequalities.

- When working with inequalities, it is useful to be aware of the following rules. Here a, b, c, d are (real) numbers.

1. If $a < b$ then $a + c < b + c$.
2. If $a < b$ and $c < d$ then $a + c < b + d$.
3. If $a < b$ and $c > 0$ then $ac < bc$.
4. If $a < b$ and $c < 0$ then $ac > bc$. (!)
5. If $0 < a < b$ then $1/a > 1/b$. (!)

- **Solve** the inequality $3x + 1 \geqslant 4x + 2$.
- **Solution:** $1 \geqslant x + 2$ (subtract $3x$)
- $-1 \geqslant x$, i.e., $x \leqslant -1$ (subtract 2)
- Solution set: $\{x \mid x \leqslant -1\} = (-\infty, -1)$.

Example

- Intervals play a role when solving more complex inequalities.

- **Solve** the inequality $(x-2)(x-3) \leqslant 0$.
- **Solution:** $(x-2)(x-3)=0 \iff x=2$ or $x=3$.
- Removing 2 and 3 from the number line leaves 3 open intervals

$$(-\infty, 2), \qquad (2,3), \qquad (3,\infty).$$

- On each interval, we determine the signs of the two factors:

Interval	$x-2$	$x-3$	$(x-2)(x-3)$
$x<2$	$-$	$-$	$+$
$2<x<3$	$+$	$-$	$-$
$x>3$	$+$	$+$	$+$

- **Solution set:** $\{x \mid x=2 \text{ or } x=3 \text{ or } 2<x<3\} = [2,3]$.

- This method also works for more than two factors . . .

An Exam Question.

Example (MA161 Semester 1 Exam, 2013/2014.)

- **Solve** the inequality $x^2 > x + 6$.

Solution:

1. **Rewrite** the inequality as $x^2 - x - 6 > 0$.
2. Solve the **equation**: $x^2 - x - 6 = 0 \iff x = 3$ or $x = -2$.
3. **Factorize**: $x^2 - x - 6 = (x - 3)(x + 2)$.
4. **Signs** of the factors on the intervals $(-\infty, -2)$, $(-2, 3)$, $(3, \infty)$:

Interval	$x + 2$	$x - 3$	$(x + 2)(x - 3)$
$x < -2$	$-$	$-$	$+$
$-2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

5. **Solution set:** $\{x \mid x < -2$ or $x > 3\} = (-\infty, -2) \cup (3, \infty)$.

Symbols

- The following **frequently used symbols** are worth remembering:

$\mathbb{N}$	the natural numbers
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rationals
$\mathbb{R}$	the reals
$\{\ \}$	encloses sets
$\in$	is an element of
$\subseteq$	is a subset of
$<$	(strictly) less than
$\leqslant$	less than or equal
$[\ ]$	closed interval
$(\ )$	open interval

Exercises.

1. Go to the **library**. Find out where they keep the calculus books. Choose any three. Find the section where they introduce the concept of a **function**. Write down the **definition** of a function that they provide, their **explanation** of what it means, and one **example**. Rank the books in order of how useful you think they are.
2. The study of what we call "Calculus" is said to have been started by **Isaac Newton** and **Gottfried von Leibniz**. Find out **when and where they lived**, and what their major mathematical discoveries were.
3. Solve the following **inequalities**:

   (i) $4 - x^2 \leqslant 0$, (ii) $x^2 \geqslant x + 2$, (iii) $\frac{1}{2}x^2 < x^3$.
4. What **sets** are usually represented by the **symbols**

   $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$?

   For each set, determine which of the others it is a **subset** of.