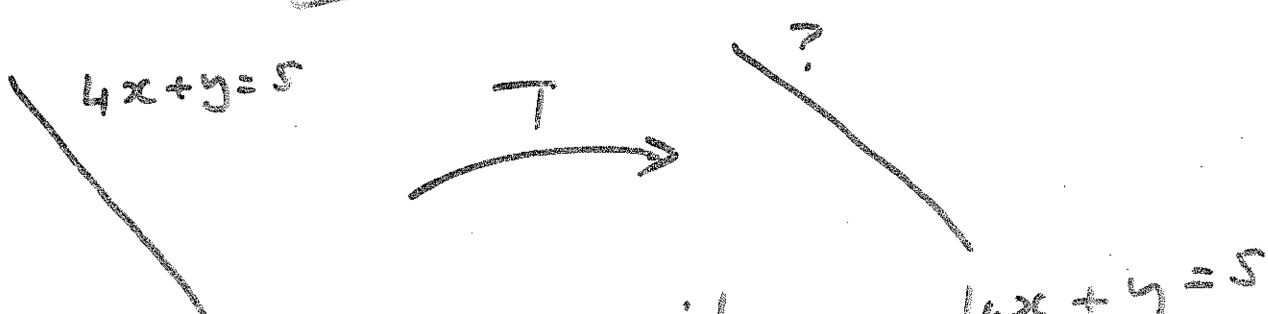
Example Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T(x,y) = (2y,\ 3x-y)$$

(i). Find the image under T of $4x+y=5$

(ii). Find a line whose image under T is $3x+2y=7$.

→ (i). $\boxed{\text{WAY I}}$



Step ① Take two points on $4x+y=5$

Take $x=0$. Then $0+y=5$ so $y=5$ $\boxed{(0,5)}$

Take $x=1$. Then $4+y=5$ so $y=1$ $\boxed{(1,1)}$

Step ② Find $T(0,5)$ and $T(1,1)$.

Note

$$T(0,5) = (2\times5,\ 3\times0-5) = (10,-5)$$

$$T(1,1) = (2\times1,\ 3\times1-1) = (2,2)$$

Step 2 Find the equation of the line L through $(10,-5)$ and $(2,2)$

$$\text{Slope of } L = \frac{2-(-5)}{2-10} = \frac{7}{-8}.$$

Equation of L is

$$y - 2 = -\frac{7}{8}(x-2)$$

i.e $\quad 8y - 16 = -7x + 14$

i.e $\quad 7x + 8y = 30.$

$\boxed{\text{WAY II}}$

Find T^{-1}

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ 3x - y \end{pmatrix} \quad \text{so} \quad A_T = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$$

Now

$$A_T^* = \begin{pmatrix} -1 & -2 \\ -3 & 0 \end{pmatrix} \quad \text{and} \quad |A_T| = 0.(-1) - 2.3 = -6$$

So

$$A_{T^{-1}} = (A_T)^{-1} = -\frac{1}{6}\begin{pmatrix} -1 & -2 \\ -3 & 0 \end{pmatrix}.$$

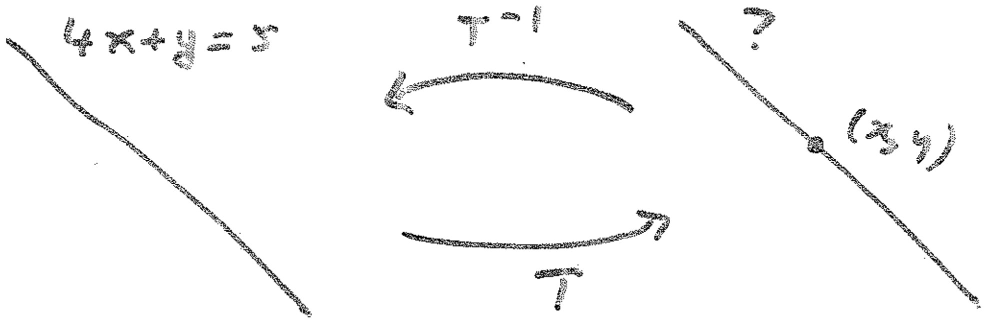
Thus

$$T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} -1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= -\frac{1}{6}\begin{pmatrix} -x - 2y \\ -3x + 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6}x + \frac{1}{3}y \\ \frac{1}{2}x \end{pmatrix},$$

i.e $T^{-1}(x,y) = \left(\frac{1}{6}x + \frac{1}{3}y, \ \frac{1}{2}x\right)$



$T^{-1}(x,y)$ is ON THE LINE $4x+y=5$

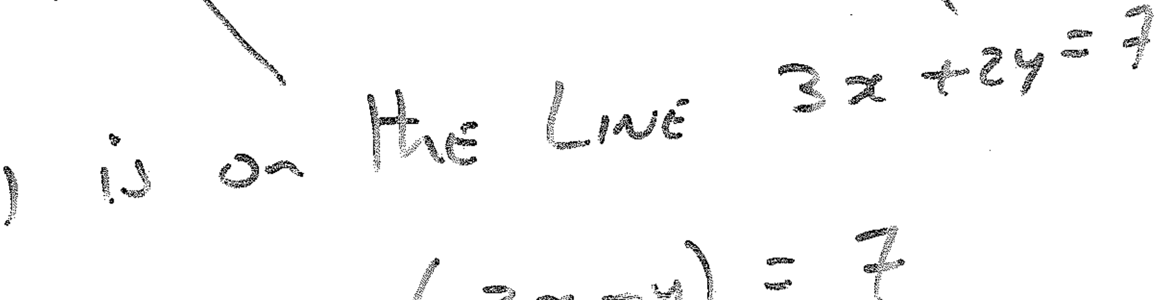
so

$$4\left(\frac{1}{6}x + \frac{1}{3}y\right) + \left(\frac{1}{2}x\right) = 5$$

i.e $\frac{7}{6}x + \frac{4}{3}y = 5$

i.e $7x + 8y = 30$.

(ii) $\boxed{\text{WAY I}}$



$T(x,y)$ is on the LINE $3x+2y=7$

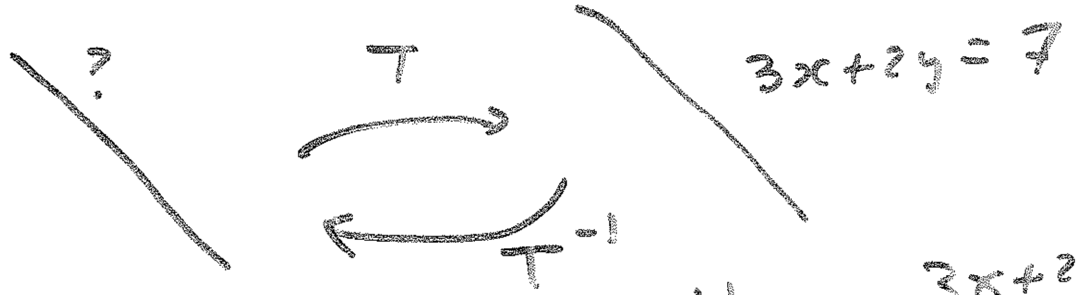
so

$$3(2y) + 2(3x-y) = 7$$

i.e $6y + 6x - 2y = 7$

i.e $6x + 4y = 7$.

WAY II



<u>STEP</u> ① Take two points on $3x+2y=7$.

Take $x=0$. Then $0+2y=7$ so $y=\frac{7}{2}$ $\boxed{(0,\frac{7}{2})}$

Take $y=0$. Then $3x+0=7$ so $x=\frac{7}{3}$ $\boxed{(\frac{7}{3},0)}$

<u>STEP</u> ② Find $T^{-1}(0,\frac{7}{2})$ and $T^{-1}(\frac{7}{3},0)$

RECALL

$$T^{-1}(x,y)=\left(\frac{1}{6}x+\frac{1}{3}y\ ,\ \frac{1}{2}x\right)$$

so

$$T^{-1}(0,\tfrac{7}{2})=\left(0+\tfrac{1}{3}(\tfrac{7}{2}),\ 0\right)=\left(\tfrac{7}{6},0\right)$$

and

$$T^{-1}(\tfrac{7}{3},0)=\left(\tfrac{1}{6}(\tfrac{7}{3})+0,\ \tfrac{1}{2}(\tfrac{7}{3})\right)=\left(\tfrac{7}{18},\tfrac{7}{6}\right)$$

<u>STEP</u> ③ Find the equation of the line L through $(\frac{7}{6},0)$ and $(\frac{7}{18},\frac{7}{6})$

Slope of L is
$$\frac{\frac{7}{6}-0}{\frac{7}{18}-\frac{7}{6}}=\frac{\frac{7}{6}}{\frac{7-21}{18}}=\frac{\left(\frac{7}{6}\right)}{\left(\frac{-14}{18}\right)}$$

$$=\frac{7}{6}\cdot\left(\frac{18}{-14}\right)=\frac{3}{-2}$$

Equation of L is

$$y - 0 = -\frac{3}{2}\left(x - \frac{7}{6}\right)$$

i.e $2y = -3\left(x - \frac{7}{6}\right)$

i.e $2y = -3x + \frac{7}{2}$

i.e $4y = -6x + 7$. //

LET $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.

NOTE $T(0) = 0$ Since

$$T(0) = T(0.0) = 0 . T(0) = 0$$

LET L be a Line through O and v ($v \neq 0$). THEN $T(L)$ is A Line through O (Since $T(0) = 0$) and $T(v)$ PROVIDED $T(v) \neq 0$.

?? WHAT HAPPENS IF $T(v)$ IS A MuLtiple of v ??

Answer THEN $T(v)$ will be on L

So the image under T of L is L itself.

A 2×2 matrix

The number λ is CALLED AN EIGENVALUE of A if there is a $v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that $Av = \lambda v$. v is CALLED AN EIGENVECTOR of A CORRESPONDING to (the eigenvalue) λ.

Let λ be an eigenvalue of A. Then there exists $v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$Av = \lambda v$$

i.e. $$Av = \lambda I v$$

i.e. $$(A - \lambda I) v = 0$$

Now since $v \neq 0$ we note

$$\det (A - \lambda I) = 0$$

λ is AN EIGENVALUE of A IF AND only IF

$$P(\lambda) = \det (A - \lambda I) = 0$$

(Characteristic polynomial of A).

LET

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The <u>TRACE</u> of A (trace(A)) is

$$\text{trace } A = a + d.$$

Note

$$A - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

$$= \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix},$$

so

$$P(\lambda) = (a-\lambda)(d-\lambda) - bc$$

$$= ad - a\lambda - d\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$= \lambda^2 - (\text{trace } A)\lambda + \det A.$$

THUS counting multiplicities every 2×2 MATRIX has exactly 2 EIGENVALUES.

Suppose λ_1, λ_2 are the eigenvalues of A. Then

$$
\begin{aligned}
P(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\
&= \lambda^2 - \lambda\lambda_2 - \lambda\lambda_1 + \lambda_1\lambda_2 \\
&= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 .
\end{aligned}
$$

So

$$\boxed{\text{trace } A = \lambda_1 + \lambda_2 \text{ and } \det A = \lambda_1\lambda_2}$$

<u>Example</u> (i). Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$$

(ii). Write down an equation of a line (through the origin) left fixed by the linear transformation defined by A.

→ <u>(i)</u> <u>One way</u>

$$0 = |A - \lambda I| = \lambda^2 - (\text{trace } A)\lambda + \det A$$

and

Now $\text{trace } A = 0 + (-1) = -1$

$\det A = 0.(-1) - 2.3 = -6.$

Thus
$$0 = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3),$$
so $\lambda = 2$, $\lambda = -3$ are the eigenvalues of A

Second way

$$0 = |A - \lambda I| = \left| \begin{pmatrix} 0-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix} \right|$$

$$= (-\lambda)(-1-\lambda) - 2 \cdot 3$$
$$= +\lambda + \lambda^2 - 6 = \lambda^2 + \lambda - 6$$
$$= (\lambda-2)(\lambda+3),$$
so $\lambda = 2$, $\lambda = -3$ are the eigenvalues of A.

For $\lambda = 2$ we must solve $(A - \lambda I)V = 0$

[i.e $\begin{pmatrix} -\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix} V = 0$] with $\lambda = 2$

i.e $\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $V = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

i.e
$$-2x_1 + 2y_1 = 0$$
$$3x_1 - 3y_1 = 0$$

Both equations say $x_1 = y_1$.

Any nonzero $V = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ with $y_1 = x_1$ is an eigenvector of A corresponding to $\lambda = 2$.

One such eigenvector is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

FOR $\lambda = -3$ WE MUST solve $(A - \lambda I)v = 0$

[i.e $\begin{pmatrix} -\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix} v = 0$] with $\lambda = -3$

i.e $\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

i.e $3x_2 + 2y_2 = 0$
$3x_2 + 2y_2 = 0$

EQUATION SAYS $y_2 = -\frac{3}{2} x_2$

Any NONZERO $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ with $y_2 = -\frac{3}{2} x_2$ is AN EIGENVECTOR of A corresponding to $\lambda = -3$. One such eigenvector is

$V_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

(ii) RECALL IF L is A Line through 0 and v $(v \neq 0)$ and $Tv = \alpha v$ then $T(L)$ is L itself.

If we take L to be the line through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} +2 \\ -3 \end{pmatrix}$ (i.e the line $y = -\frac{3}{2}x$)

then $T(L) = L$. //

LET

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

WITH EIGENVALUES λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$).

Suppose $V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ is AN EIGENVECTOR of A corresponding to λ_1 and $V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is an eigenvector of A corresponding to λ_2.

LET

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad E = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

ONE CAN show

$$AE = ED$$

So IF $|E| \neq 0$ then

$$A = EDE^{-1}.$$

NOTE

$$A^2 = AA = EDE^{-1}EDE^{-1} = EDIDE^{-1}$$

$$= ED^2E^{-1},$$

$$A^3 = AA^2 = EDE^{-1}ED^2E^{-1}$$
$$= EDID^2E^{-1} = ED^3E^{-1},$$

$\vdots$

So

$$\boxed{A^n = ED^nE^{-1} \quad \text{for } n \in \{1, 2, \ldots\}}$$

NOTE

$$D^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + 0 & 0 + 0 \\ 0 + 0 & 0 + \lambda_2^2 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix},$$

$$D^3 = D\,D^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 + 0 & 0 + 0 \\ 0 + 0 & 0 + \lambda_2^3 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix},$$

$\vdots$

So

$$\boxed{D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \quad \text{for } n \in \{1, 2, \ldots\}}$$

Example

Let $A = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$

(a). Find the eigenvalues and eigenvectors of A.

(b). Find a diagonal matrix D and an invertible matrix E such that $AE = ED$.

(c). Calculate A^{100}

(d). Express the determinant of A in terms of the eigenvalues of A

$\rightarrow$ (a) One way

$0 = |A - \lambda I| = \lambda^2 - (\text{trace } A)\lambda + \det A$

Now trace $A = 2 + 4 = 6$ and

$\det A = 2.4 - [(-1)(-3)] = 8 - 3 = 5$

Thus

$$0 = \lambda^2 - 6\lambda + 5$$
$$= (\lambda - 5)(\lambda - 1),$$

So $\lambda = 1$, $\lambda = 5$ are the eigenvalues of A.

Second way

$$0 = |A - \lambda I| = \left| \begin{pmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{pmatrix} \right|$$

$= (2-\lambda)(4-\lambda) - 3$

$= 8 - 2\lambda - 4\lambda + \lambda^2 - 3$

$= \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5)$

So $\lambda=1$, $\lambda=5$ are the eigenvalues of A.

For $\lambda=1$ we must solve $(A-\lambda I)v = 0$

[i.e $\begin{pmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{pmatrix} v = 0$] with $\lambda = 1$

i.e $\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$]

i.e $\quad x_1 - y_1 = 0$
$\quad -3x_1 + 3y_1 = 0$

Both equations say $x_1 = y_1$

Any nonzero $v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ with $y_1 = x_1$ is an eigenvector of A corresponding to $\lambda = 1$.

One such eigenvector is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = 5$ we must solve $(A-\lambda I)v = 0$

[i.e $\begin{pmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{pmatrix} v = 0$] with $\lambda = 5$

i.e $\begin{pmatrix} -3 & -1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

i.e $\quad -3x_2 - y_2 = 0$
$\quad -3x_2 - y_2 = 0$

Equation says $y_2 = -3x_2$.

Any nonzero $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ with $y_2 = -3x_2$

is an eigenvector of A corresponding to $\lambda = 5$. One such eigenvector is $V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

(b) $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$

(c) $A^{100} = E D^{100} E^{-1}$.

Note $E^* = \begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}$ and $\det E = 1.(-3) - 1.1 = -3 - 1 = -4$ so

$$E^{-1} = -\frac{1}{4}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}.$$

Also note

$$D^{100} = \begin{pmatrix} 1^{100} & 0 \\ 0 & 5^{100} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5^{100} \end{pmatrix}$$

Thus

$$A^{100} = E D^{100} E^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^{100} \end{pmatrix}\left(-\frac{1}{4}\right)\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}$$
$$= -\frac{1}{4}\begin{pmatrix} 1 & 5^{100} \\ 1 & -3 \times 5^{100} \end{pmatrix}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}$$
$$= -\frac{1}{4}\begin{pmatrix} -3 - 5^{100} & -1 + 5^{100} \\ -3 + 3 \times 5^{100} & -1 - 3 \times 5^{100} \end{pmatrix}$$

(d) <u>WAY 1</u>

$$|A| = \lambda_1 \cdot \lambda_2 = 1.5$$

<u>WAY 2</u>

$A = EDE^{-1}$ so

$$|A| = |EDE^{-1}| = |E|\,|D|\,|E^{-1}|$$

$$= |E|\,|D|\,\frac{1}{|E|}$$

$$= |D|$$

$$= 1.5 - 0 \quad \text{since } D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= 1.5. \quad //$$

<u>Example</u> Let $A = \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}$.

<u>(i)</u> Find the eigenvalues and eigenvectors of A.

<u>(ii)</u>. Write down a matrix E and a diagonal matrix D such that $AE = ED$. Now calculate A^n for $n \in \mathbb{N} = \{1, 2, \dots\}$.

<u>(iii)</u> Solve the recurrence relation (i.e. find x_n, y_n for $n \in \mathbb{N}$)

$$x_{n+1} = -4x_n + y_n$$

$$y_{n+1} = 4x_n - 4y_n$$

→ (i) One WAY

$0 = |A - \lambda I| = \lambda^2 - (\text{trace } A)\lambda + \det A.$

Now trace $A = -4 + (-4) = -8$ and

$\det A = (-4).(-4) - 1.4 = 16 - 4 = 12.$

THUS

$0 = \lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6)$

So $\lambda = -2, \lambda = -6$ are the eigenvalues of A.

SECOND WAY

$$0 = |A - \lambda I| = \left| \begin{pmatrix} -4-\lambda & 1 \\ 4 & -4-\lambda \end{pmatrix} \right|$$

$$= (-4-\lambda)(-4-\lambda) - 4$$
$$= 16 + 4\lambda + 4\lambda + \lambda^2 - 4$$
$$= \lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6)$$

So $\lambda = -2, \lambda = -6$ are the eigenvalues of A.

For $\lambda = -2$ WE MUST solve $(A - \lambda I)V = 0$

[i.e $\begin{pmatrix} -4-\lambda & 1 \\ 4 & -4-\lambda \end{pmatrix} V = 0$] with $\lambda = -2$

i.e $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $V = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

i.e $-2x_1 + y_1 = 0$
$4x_1 - 2y_1 = 0$

BOTH EQUATIONS SAY $y_1 = 2x_1$.

ANY NONZERO $V = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ with $y_1 = 2x_1$

is An EIGENVECTOR OF A Corresponding to $\lambda = -2$

One such eigenvector is $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $\lambda = -6$ we must solve $(A - \lambda I)v = 0$

[i.e $\begin{pmatrix} -4-\lambda & 1 \\ 4 & -4-\lambda \end{pmatrix} v = 0$] with $\lambda = -6$

i.e $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

i.e $\quad 2x_2 + y_2 = 0$
$\quad\quad 4x_2 + 2y_2 = 0$

Both equations say $y_2 = -2x_2$.

Any nonzero $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ with $y_2 = -2x_2$ is an eigenvector of A corresponding to $\lambda = -6$.

One such eigenvector is $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(ii) $\quad D = \begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$

Note $E^* = \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$ and $\det E = 1.(-2) - 1.2$

$= -2 - 2 = -4$ so

$$E^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}.$$

Also note

$$D^n = \begin{pmatrix} (-2)^n & 0 \\ 0 & (-6)^n \end{pmatrix}$$

So

$$A^n = E D^n E^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & (-6)^n \end{pmatrix} \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$ (38)

$$= \frac{1}{4} \begin{pmatrix} (-2)^n & (-6)^n \\ 2(-2)^n & -2(-6)^n \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2(-2)^n + 2(-6)^n}{4} & \frac{(-2)^n - (-6)^n}{4} \\ \frac{4(-2)^n - 4(-6)^n}{4} & \frac{2(-2)^n + 2(-6)^n}{4} \end{pmatrix}.$$

iii)
$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

NOTE

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = A \begin{pmatrix} x_{m-1} \\ y_{m-1} \end{pmatrix} \qquad \underline{n = m-1}$$

$$= A \cdot A \begin{pmatrix} x_{m-2} \\ y_{m-2} \end{pmatrix} = A^2 \begin{pmatrix} x_{m-2} \\ y_{m-2} \end{pmatrix} \qquad \underline{n = m-2}$$

$$= A^2 A \begin{pmatrix} x_{m-3} \\ y_{m-3} \end{pmatrix} = A^3 \begin{pmatrix} x_{m-3} \\ y_{m-3} \end{pmatrix} \qquad \underline{n = m-3}$$

$$= \vdots$$

$$= A^m \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

THUS

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \frac{2(-2)^n + 2(-6)^n}{4} & \frac{(-2)^n - (-6)^n}{4} \\ \frac{4(-2)^n - 4(-6)^n}{4} & \frac{2(-2)^n + 2(-6)^n}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2(-2)^n + 2(-6)^n + (-2)^n - (-6)^n}{4} \\ \frac{4(-2)^n - 4(-6)^n + 2(-2)^n + 2(-6)^n}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3(-2)^n + (-6)^n}{4} \\ \frac{6(-2)^n - 2(-6)^n}{4} \end{pmatrix}$$

So $x_n = \frac{3}{4}(-2)^n + \frac{1}{4}(-6)^n$ and $y_n = \frac{3}{2}(-2)^n - \frac{1}{2}(-6)^n$.

Example LET $A = \begin{pmatrix} 5 & -4 \\ 1 & 0 \end{pmatrix}$

(i) FIND the eigen values and eigenvectors of A

(ii) Find A^n for $n \in \mathbb{N} = \{1, 2, ..\}$

(iii) Solve the DIFFERENCE equation (i.e find x_n for $n \in \mathbb{N}$)

$$x_{n+1} = 5x_n - 4x_{n-1} \quad \text{for } n \in \mathbb{N}$$

given that $x_0 = 5$ and $x_1 = 17$.

(i)

One WAY

$0 = |A - \lambda I| = \lambda^2 - (\text{trace } A)\lambda + \det A$

Now $\text{trace } A = 5 + 0 = 5$ and $\det A = 5.0 - ((-4).1)$

$= 0 + 4 = 4.$

THUS

$0 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$

So $\lambda = 1$, $\lambda = 4$ ARE the eigenvalues of A.

Second WAY

$$0 = |A - \lambda I| = \left| \begin{pmatrix} 5-\lambda & -4 \\ 1 & -\lambda \end{pmatrix} \right|$$

$$= (5-\lambda)(-\lambda) - (-4)$$

$$= -5\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4),$$

So $\lambda = 1$, $\lambda = 4$ Are the eigenvalues of A.

For $\lambda = 1$ WE MUST solve $(A - \lambda I)V = 0$

[i.e $\begin{pmatrix} 5-\lambda & -4 \\ 1 & -\lambda \end{pmatrix} V = 0$] with $\lambda = 1$

i.e $\begin{pmatrix} 4 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $V = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

i.e $4x_1 - 4y_1 = 0$
$x_1 - y_1 = 0$

BOTH EQUATIONS SAY $y_1 = x_1$

Any nonzero $v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ with $x_1 = y_1$ is an eigenvector of A corresponding to $\lambda = 1$. One such eigenvector is $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = 4$ we must solve $(A - \lambda I)v = 0$

[i.e $\begin{pmatrix} 5-\lambda & -4 \\ 1 & -\lambda \end{pmatrix} v = 0$] with $\lambda = 4$.

i.e $\begin{pmatrix} 1 & -4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ where $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

i.e $x_2 - 4y_2 = 0$

Equation says $x_2 = 4y_2$

Any nonzero $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ with $x_2 = 4y_2$ is an eigenvector of A corresponding to $\lambda = 4$. One such eigenvector is $v = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

(ii). $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

Note $E^* = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix}$ and $\det E = 1 - 4 = -3$

so $E^{-1} = \frac{1}{-3}\begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix}$. Also $D^n = \begin{pmatrix} 1^n & 0 \\ 0 & 4^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix}$

Now

$$A^n = E D^n E^{-1} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix} \frac{1}{-3}\begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 1 & 4^{n+1} \\ 1 & 4^n \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1-4^{n+1} & -4+4^{n+1} \\ 1-4^n & -4+4^n \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{-1+4^{n+1}}{3} & \dfrac{4-4^{n+1}}{3} \\ \dfrac{-1+4^n}{3} & \dfrac{4-4^n}{3} \end{pmatrix}$$

(iii) $x_{n+1} = 5x_n - 4x_{n-1}$

$x_n = 1x_n + 0x_{n-1}$

So

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = A\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$$

Now

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \qquad \underline{n=1}$$

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = A^2\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \qquad \underline{n=2}$$

$$\begin{pmatrix} x_4 \\ x_3 \end{pmatrix} = A\begin{pmatrix} x_3 \\ x_2 \end{pmatrix} = A^3\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \qquad \underline{n=3}$$

$$\vdots$$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^n\begin{pmatrix} x_1 \\ x_0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \dfrac{-1+4^{n+1}}{3} & \dfrac{4-4^{n+1}}{3} \\ \dfrac{-1+4^n}{3} & \dfrac{4-4^n}{3} \end{pmatrix}\begin{pmatrix} 17 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{-17+17.4^{n+1}+20-5.4^{n+1}}{3} \\ \dfrac{-17+17.4^n+20-5.4^n}{3} \end{pmatrix}.$$

In particular

$$X_n = \frac{-17 + 17.6^n + 20 - 5.6^n}{3}$$

$$= \frac{3 + 12.6^n}{3} = 1 + 4.6^n = 1 + 6^{n+1}.$$

Alternate approach for (iii)

Note $X_n = C_1 \lambda_1^n + C_2 \lambda_2^n$

so $X_n = C_1 + C_2 4^n$

Now $x_1 = 17$, $x_0 = 5$ so

$$5 = C_1 + C_2$$

and $$17 = C_1 + 4C_2$$

Thus $12 = 3C_2$ i.e $C_2 = 4$

and $5 = C_1 + 4$ i.e $C_1 = 1$

Thus $X_n = 1 + 4.4^n.$