2 × 2 MATRICES

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

α REAL NUMBER

① $$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

② $$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$$

③ $$A^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

TRANSPOSE OF A

[OBTAINED BY INTERCHANGING THE ROWS AND COLUMNS OF A.]

TWO SPECIAL MATRICES

ZERO MATRIX $$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

IDENTITY MATRIX $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2 × 2 MATRICES

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

α REAL NUMBER

(1) $$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

(2) $$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$$

(3) $$A^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

TRANSPOSE of A

[OBTAINED by INTERCHANGING the ROWS And columns of A.]

TWO SPECIAL MATRICES

ZERO MATRIX $$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

IDENTITY MATRIX $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2)

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & 7 \\ 3 & -2 \end{pmatrix}$$

FIND $4A$, $A+B$, $5A+3B-6I$, $3A^t$, $5A-4A^t$.

$$\rightarrow 4A = 4\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -4 & 20 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 1+6 & 2+7 \\ (-1)+3 & 5+(-2) \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 2 & 3 \end{pmatrix}$$

$$5A+3B-6I = \begin{pmatrix} 5 & 10 \\ -5 & 25 \end{pmatrix} + \begin{pmatrix} 18 & 21 \\ 9 & -6 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 5+18+(-6) & 10+21+0 \\ (-5)+9+0 & 25+(-6)+(-6) \end{pmatrix} = \begin{pmatrix} 17 & 31 \\ 4 & 13 \end{pmatrix}$$

$$3A^t = 3\begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 6 & 15 \end{pmatrix}$$

$$5A-4A^t = \begin{pmatrix} 5 & 10 \\ -5 & 25 \end{pmatrix} + (-4)\begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 10 \\ -5 & 25 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ -8 & -20 \end{pmatrix}$$

$$= \begin{pmatrix} 5+(-4) & 10+4 \\ (-5)+(-8) & 25+(-20) \end{pmatrix} = \begin{pmatrix} 1 & 14 \\ -13 & 5 \end{pmatrix} //$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

THEN

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{pmatrix}$$

AND

$$BA = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} a_{11} + b_{12} a_{21} & b_{11} a_{12} + b_{12} a_{22} \\ b_{21} a_{11} + b_{22} a_{21} & b_{21} a_{12} + b_{22} a_{22} \end{pmatrix}.$$

$A_{2\times 2}$ $\qquad$ $B_{2\times 2}$

$$AB = \begin{pmatrix} \underset{(A)}{\text{Row } 1} \times \underset{(B)}{\text{Column } 1} & \underset{(A)}{\text{Row } 1} \times \underset{(B)}{\text{Column } 2} \\ \underset{(A)}{\text{Row } 2} \times \underset{(B)}{\text{Column } 1} & \underset{(A)}{\text{Row } 2} \times \underset{(B)}{\text{Column } 2} \end{pmatrix}$$

$\times$ means dot product.

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix}$$

Find $3AB + 2A^2$ and $5A^3$.

→ To find $3AB + 2A^2$ we first find AB and A^2.

NOTE

$$AB = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3+15 & -2+18 \\ -6+20 & 4+24 \end{pmatrix} = \begin{pmatrix} 18 & 16 \\ 14 & 28 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & 3+12 \\ -2-8 & -6+16 \end{pmatrix} = \begin{pmatrix} -5 & 15 \\ -10 & 10 \end{pmatrix}$$

so

$$3AB + 2A^2 = 3\begin{pmatrix} 18 & 16 \\ 14 & 28 \end{pmatrix} + 2\begin{pmatrix} -5 & 15 \\ -10 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 54 & 48 \\ 42 & 84 \end{pmatrix} + \begin{pmatrix} -10 & 30 \\ -20 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} 44 & 78 \\ 22 & 104 \end{pmatrix}.$$

To find $5A^3$ we first find A^3

Note

$$A^3 = A A^2 \quad [= A.A.A = A^2 A]$$

$$= \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} -5 & 15 \\ -10 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} -5-30 & 15+30 \\ 10-40 & -30+40 \end{pmatrix}$$

$$= \begin{pmatrix} -35 & 45 \\ -30 & 10 \end{pmatrix}$$

So

$$5A^3 = 5 \begin{pmatrix} -35 & 45 \\ -30 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} -175 & 225 \\ -150 & 50 \end{pmatrix}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The <u>ADJOINT</u> of A (A^*) is

$$A^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

The <u>DETERMINANT</u> of A ($|A|$ or $\det(A)$)

is

$$|A| = a_{11}\,a_{22} - a_{12}\,a_{21}.$$

The <u>INVERSE</u> of A (A^{-1}) [provided $|A| \neq 0$] is

$$A^{-1} = \frac{1}{|A|}\,A^* = \frac{1}{[a_{11}a_{22} - a_{12}a_{21}]}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

<u>NOTE</u>

$$A\,A^* = A^*\,A = (\det A)\,I$$

So if $|A| \neq 0$,

$$A\,A^{-1} = A^{-1}A = I.$$

A , B 2 × 2 MATRICES

If $AB = BA = I$
then B is called the <u>INVERSE</u> of A

<u>More Information</u>

C, D 2 × 2 MATRICES
α a REAL Number.

NOTE
$$|CD| = |C|\,|D|$$
$$|C^t| = |C|$$
$$|C^*| = |C|$$
$$|\alpha C| = \alpha^2 |C|$$

The INVERSE of C EXISTS IF AND only IF $|C| \neq 0$.

If $|C| \neq 0$ then
$$|C^{-1}| = \frac{1}{|C|}.$$

EXAMPLE LET

$$A = \begin{pmatrix} 1 & -3 \\ -4 & 2 \end{pmatrix}.$$

FIND A^*, $|A|$ and A^{-1}.

$\longrightarrow$ $$A^* = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$|A| = 1 \cdot 2 - [(-3) \cdot (-4)]$$
$$= 2 - [12] = -10$$

$$A^{-1} = \frac{1}{(-10)} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{-10} & \frac{3}{-10} \\ \frac{4}{-10} & \frac{1}{-10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} & -\frac{3}{10} \\ -\frac{2}{5} & -\frac{1}{10} \end{pmatrix}.$$

Example Suppose

$$A = \begin{pmatrix} -2 & 1 \\ 5 & -4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}.$$

Find $(AB)^{-1}$.

$$\to \quad AB = \begin{pmatrix} -2 & 1 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -6+1 & -8+2 \\ 15-4 & 20-8 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -6 \\ 11 & 12 \end{pmatrix}.$$

Now

$$(AB)^* = \begin{pmatrix} 12 & 6 \\ -11 & -5 \end{pmatrix}$$

and

$$|AB| = (-5)(12) - [(-6)\,11] = -60 - [-66]$$

$$= -60 + 66 = 6.$$

Thus

$$(AB)^{-1} = \frac{1}{6}\begin{pmatrix} 12 & 6 \\ -11 & -5 \end{pmatrix}.$$

More Information

A, B 2×2 matrices with $|A| \neq 0$, $|B| \neq 0$.

Then $(AB)^{-1} = B^{-1} A^{-1}$

2 x 2 MATRIX

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

2 x 1 MATRIX

$$B = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{pmatrix}.$$

(2 x 1 MATRIX)

CONSIDER THE SYSTEM

$$① \begin{cases} a_{11}x_1 + a_{12}x_2 = u \\ a_{21}x_1 + a_{22}x_2 = v \end{cases}$$

WHERE $a_{11}, a_{12}, a_{21}, a_{22}, u, v$ ARE KNOWN and $a_{11}a_{22} - a_{12}a_{21} \neq 0$. FIND x_1, x_2 which Satisfy ①.

STEP ① LET

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } b = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Now ① can be written as

$$Ax = b$$

STEP ② The solution is $x = A^{-1}b$.

$\rightarrow Ax = b$ so $A^{-1}Ax = A^{-1}b$

i.e

$$I x = A^{-1} b$$

i.e $\quad x = A^{-1} b \quad$ Since

$$I x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 0 \\ 0 + x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x.$$

<u>Example</u> Solve

2

$$\begin{cases} x_1 - 3x_2 = 4 \\ -4x_1 + 2x_2 = 6 \end{cases}$$

→ LET

$$A = \begin{pmatrix} 1 & -3 \\ -4 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}.$$

Now 2 can be rewritten as $Ax = b$ so

$x = A^{-1} b$.

Note

$$A^{*} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, \quad |A| = 1.2 - [(-3)(-4)] = 2 - 12 = -10$$

so

$$A^{-1} = -\frac{1}{10} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}.$$

THUS

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{1}{10} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix} = -\frac{1}{10} \begin{pmatrix} 8 + 18 \\ 16 + 6 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{26}{10} \\ -\frac{22}{10} \end{pmatrix} = \begin{pmatrix} -\frac{13}{5} \\ -\frac{11}{5} \end{pmatrix}$$

i.e $\quad x_1 = -\frac{13}{5}, \quad x_2 = -\frac{11}{5}.$

A <u>linear transformation</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a function that assigns to each $v \in \mathbb{R}^2$ a unique $Tv \in \mathbb{R}^2$ and satisfies

$$T(u+v) = T(u) + T(v)$$

and

$$T(\alpha u) = \alpha T(u)$$

for each $u, v \in \mathbb{R}^2$ and α a real number

<u>MORE Information</u>

$T: \mathbb{R}^2 \to \mathbb{R}^2$ a LINEAR transformation THEN THERE EXISTS A unique 2×2 Matrix A_T (CALLED the <u>NATURAL MATRIX</u>) such that

$$Tx = A_T x \quad \text{for EVERY } x \in \mathbb{R}^2.$$

LET $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and

$$Tx = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

[we ALSO write it as
$T(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 \,,\, a_{21}x_1 + a_{22}x_2)$]

Note

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So

$$A_T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Note

$$A_T = \begin{pmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

Since

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11}+0 \\ a_{21}+0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

and

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+a_{12} \\ 0+a_{22} \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

<u>Example</u> Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+2y \\ 3x+4y \end{pmatrix}$$

[or $T(x,y) = (-x+2y,\ 3x+4y)$].

Find its natural matrix A_T.

$$\longrightarrow \quad A_T = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}.$$

<u>Example</u> The linear transformation $S: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$A_S = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}.$$

Find $S(x, y)$.

$$\to \quad S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 3x + 1\cdot y \\ -2x + 4\cdot y \end{pmatrix}$$

i.e.

$$S(x, y) = (3x + y \ , \ -2x + 4y).$$

<u>Example</u> The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(2, 1) = (1, -1)$ and $T(1, -3) = (2, 0)$. Find the natural matrix A_T and $T(x, y)$.

$$\to \quad \text{Assume} \quad A_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{i.e.} \quad \begin{array}{l} 2a + b = 1 \\ 2c + d = -1 \end{array}$$

(ii

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \underline{\text{i.e}} \quad \begin{array}{l} a-3b=2 \\ c-3d=0 \end{array}$$

THUS

$$\begin{array}{l} 2a+b=1 \\ a-3b=2 \end{array} \quad \text{so} \quad \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

AND

$$\begin{array}{l} 2c+d=-1 \\ c-3d=0 \end{array} \quad \text{so} \quad \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

LET

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \quad \text{and note} \quad A^{-1} = -\frac{1}{7}\begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix}$$

Thus

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} \\ -\frac{3}{7} \end{pmatrix}$$

and

$$\begin{pmatrix} c \\ d \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} \\ -\frac{1}{7} \end{pmatrix}.$$

THUS

$$\begin{array}{l} a = \frac{5}{7} \\ b = -\frac{3}{7} \\ c = -\frac{3}{7} \end{array}$$

So

$$A_T = \begin{pmatrix} \frac{5}{7} & -\frac{3}{7} \\ -\frac{3}{7} & -\frac{1}{7} \end{pmatrix}.$$

Also

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & -\frac{3}{7} \\ -\frac{3}{7} & -\frac{1}{7} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{7}x - \frac{3}{7}y \\ -\frac{3}{7}x - \frac{1}{7}y \end{pmatrix}$$

i.e

$$T(x,y) = \left(\frac{5}{7}x - \frac{3}{7}y, -\frac{3}{7}x - \frac{1}{7}y\right). //$$

Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations.

Define $T_1 \circ T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$(T_1 \circ T_2)(u) = T_1(T_2(u))$ for $u \in \mathbb{R}^2$.

Now $T_1 \circ T_2$ is a linear transformation and $A_{T_1} A_{T_2}$ is the natural matrix associated with $T_1 \circ T_2$ i.e $A_{T_1 \circ T_2} = A_{T_1} A_{T_2}$

Suppose $T_1^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ exists (i.e suppose $|A_{T_1}| \neq 0$). Note

$$T_1 \circ T_1^{-1} = T_1^{-1} \circ T_1 = I.$$

Now T_1^{-1} is a LINEAR transformation and $(A_{T_1})^{-1}$ is the natural matrix associated with T_1^{-1} i.e

$$A_{T_1^{-1}} = (A_{T_1})^{-1}.$$

<u>Example</u> LET $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations defined by

$$T(x, y) = (5x - 3y, \; 7x + 2y)$$

and

$$S(x, y) = (2x + 2y, \; -3x + 8y).$$

Find the natural matrices of the linear transformations

$$5T + 3S, \quad T \circ S, \quad T^{-1}$$

and find

$$(5T + 3S)(x, y), \quad T \circ S(x, y) \text{ and } T^{-1}(x, y)$$

→ Now

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - 3y \\ 7x + 2y \end{pmatrix}, \quad S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+2y \\ -3x+8y \end{pmatrix}$$

so

$$A_T = \begin{pmatrix} 5 & -3 \\ 7 & 2 \end{pmatrix} \text{ and } A_S = \begin{pmatrix} 2 & 2 \\ -3 & 8 \end{pmatrix}$$

Now

$$A_{5T} = 5A_T = 5\begin{pmatrix} 5 & -3 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 25 & -15 \\ 35 & 10 \end{pmatrix}$$

and

$$A_{3S} = 3\,A_S = 3\begin{pmatrix} 2 & 2 \\ -3 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ -9 & 24 \end{pmatrix}.$$

Thus

$$A_{5T+3S} = A_{5T} + A_{3S} = \begin{pmatrix} 25 & -15 \\ 35 & 10 \end{pmatrix} + \begin{pmatrix} 6 & 6 \\ -9 & 24 \end{pmatrix}$$

$$= \begin{pmatrix} 31 & -9 \\ 26 & 34 \end{pmatrix}$$

so

$$(5T+3S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 31 & -9 \\ 26 & 34 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 31x - 9y \\ 26x + 34y \end{pmatrix}.$$

Next

$$A_{T\circ S} = A_T A_S = \begin{pmatrix} 5 & -3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -3 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 10+9 & 10-24 \\ 14-6 & 14+16 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & -14 \\ 8 & 30 \end{pmatrix}$$

so

$$(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 19 & -14 \\ 8 & 30 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 19x - 14y \\ 8x + 30y \end{pmatrix}.$$

ALTERNATE Approach

$$T \circ S(x, y) = T(S(x, y))$$

$$= T(2x + 2y, -3x + 8y)$$

$$= (5(2x+2y) - 3(-3x+8y), 7(2x+2y) + 2(-3x+8y))$$

$$= (10x + 10y + 9x - 24y, 14x + 14y - 6x + 16y)$$

$$= (19x - 14y, 8x + 30y).$$

Finally

$$A_{T^{-1}} = (A_T)^{-1} \text{ where } A_T = \begin{pmatrix} 5 & -3 \\ 7 & 2 \end{pmatrix}$$

Now $A_T^* = \begin{pmatrix} 2 & 3 \\ -7 & 5 \end{pmatrix}$ and $|A_T| = 5.2 - [(-3)7]$

$$= 10 - [-21]$$

$$= 10 + 21 = 31.$$

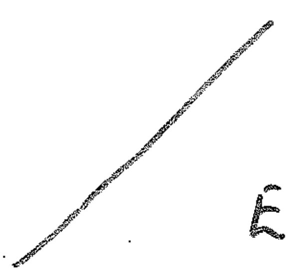
Thus

$$A_{T^{-1}} = \frac{1}{31}\begin{pmatrix} 2 & 3 \\ -7 & 5 \end{pmatrix}$$

so

$$T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 2 & 3 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{2x+3y}{31} \\[2ex] \dfrac{-7x+5y}{31} \end{pmatrix}.$$



points V_1, V_2 on L

Equation of L is

$t V_1 + (1-t) V_2 \quad , t \in \mathbb{R}$

LET $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear transformation

Now $T(tV_1 + (1-t)V_2) = t\,T(V_1) + (1-t)\,T(V_2)$

so

$T(L)$ is A LINE THROUGH $T(V_1)$ and $T(V_2)$ Provided $T(V_1) \neq T(V_2)$