Problem Sheet 3

1. For each of the following matrices A:
   (i) find the eigenvalues and eigenvectors of A;
   (ii) write down a matrix E and a diagonal matrix D, such that $AE = ED$;
   (iii) check that $A = EDE^{-1}$, and deduce that $A^n = ED^nE^{-1}$;
   (iv) hence calculate A^7.

   (a) $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 5/2 \\ -1 & -3/2 \end{pmatrix}$

2. Let A be the matrix $\begin{pmatrix} 4 & 2 \\ 5 & 7 \end{pmatrix}$.
   (a) Find the eigenvalues and eigenvectors of A. Hence write down a line whose image under A is itself.
   (b) Write down matrices E and D, with D diagonal, such that $AE = ED$ and hence calculate A^6.

3. Use the Principle of Induction to prove the following equations for all $n \geq 1$:
   (a) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$;
   (c) $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -3n \\ 0 & 1 \end{pmatrix}$; (d) $1+4+7+\ldots+(3n-2) = \frac{1}{2}n(3n-1)$;
   (e) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$;
   (f) $1 + 5 + 5^2 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$; (g) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$;
   (h) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

4. Use the Principle of Induction to prove the following statements for all $n \geq 1$:
   (a) $7^n - 1$ is divisible by 6; (b) $4^{2n} - 1$ is divisible by 15;
   (c) $n^3 + 2n$ is divisible by 3; (d) $x^{2n} - y^{2n}$ is divisible by $x + y$.

5. Prove by induction that
   (i) 3 divides $7^n - 1$ for $n \geq 1$; (ii) $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix}$ for $n \geq 1$.