Problem Sheet 2

1. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation with $T(1,3) = (0,-1)$ and $T(2,-1) = (3,2)$. Find the matrix for T.

2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by

$$T(x,y) = (-7x + 2y\,,\, 6x + 5y).$$

(i) Find the natural matrix of the linear transformations T^{-1} and find $T^{-1}(x,y)$.

(ii). Find the image of the line $3x + 4y = 12$ under T.

3. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ and $S : \mathbf{R}^2 \to \mathbf{R}^2$ be linear transformations defined by

$$T(x,y) = (5x - 2y\,,\, x + 2y) \quad \text{and} \quad S(x,y) = (-3x + 2y\,,\, 4x + y).$$

(i). Find the image of the line $2x + 5y = 10$ under T.

(ii). Find the line whose image under T is $2x + 3y = 6$.

(iii). Find the natural matrices of the linear transformations $T \circ S$ and T^{-1}.

4. Let

$$A = \begin{pmatrix} 6 & 5 \\ -3 & -2 \end{pmatrix}.$$

(i). Find all the eigenvalues of A.

(ii). Find an eigenvector of A corresponding to each eigenvalue.

(iii). Find a diagonal matrix D and an invertible matrix E such that $A\,E = E\,D$.

5. Let

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix}.$$

(i). Find all the eigenvalues of A.

(ii). Find an eigenvector of A corresponding to each eigenvalue.

(iii). Solve the recurrence relation (here $n \in \{0, 1, 2, \ldots\}$)

$$\begin{cases} x_{n+1} = 5x_n + y_n \\ y_{n+1} = -2x_n + 2y_n \end{cases}$$

given that $x_0 = 2$ and $y_0 = 3$.