SPECIAL INVOLUTIONS AND BULKY PARABOLIC SUBGROUPS IN FINITE COXETER GROUPS

GÖTZ PFEIFFER AND GERHARD RÖHRLE

1. Introduction

In [3] Felder and Veselov considered the standard and twisted actions of a finite Coxeter group $W$ on the cohomology $H^*(M_W)$ of the complement of the complexified hyperplane arrangement $M_W$ of $W$. The twisted action is obtained by combining the standard action with complex conjugation; we refer the reader to [3] for precise statements. In a case by case argument, Felder and Veselov obtain a formula for all Coxeter groups $W$ for the standard action

$$H^*(M_W) \cong \sum_{\sigma \in X_W} (2 \cdot 1^W_{\langle \sigma \rangle} - \varrho)$$

as $CW$-modules, where $X_W$ is a set of representatives of $W$-conjugacy classes of so called special involutions in $W$, $\varrho$ is the regular representation of $W$, and $1^W_{\langle \sigma \rangle}$ is the $CW$-module induced from the trivial $C\langle \sigma \rangle$-module. This formula can be deduced from earlier work of Lehrer [8, 9] and Fleischmann-Janiszczak [4, 5]. The main contribution in [3] to the theory is a uniform geometric description of the sets $X_W$ of $W$-conjugacy classes of special involutions used in the formula above.

Felder and Veselov give a similar formula for the twisted action where the summation is taken over the set of even elements from $X_W$.

In this note we give a short intrinsic characterisation of special involutions in terms of bulky parabolic subgroups.

2. Notation and Preliminaries

Throughout, $W$ denotes a finite Coxeter group, generated by a set of simple reflections $S \subseteq W$; see [1] or [6] for a general introduction into the theory of Coxeter groups. For $J \subseteq S$, let $W_J$ be the parabolic subgroup of $W$ generated by $J$ and $w_J$ denotes the unique word in $W_J$ of maximal length (with respect to $S$). Let $T = S^W$ be the set of all reflections of $W$. Let $\Phi$ be a root system with Coxeter group $W$ and $\Phi_J$
is the root subsystem of $\Phi$ corresponding to $W_J$. Set $V := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. Then $V$ affords the usual reflection representation of $W$. For each involution $\sigma \in W$ we have a direct sum decomposition $V = V_1 \oplus V_{-1}$, where $V_1$ and $V_{-1}$ are the 1 and $-1$-eigenspaces of $V$ of $\sigma$, respectively. For $\epsilon = \pm 1$ let $\Phi_\epsilon := \Phi \cap V_\epsilon$. Note that for $s = w_J$ we have $\Phi_{-1} = \Phi_J$.

Following [3], we say that an involution $\sigma$ in $W$ is special, if for any root $\alpha \in \Phi$ at least one of its projections onto $V_\epsilon$ is proportional to a root in $\Phi_\epsilon$. Clearly, this definition does not depend on the choice of root system for $W$.

The conjugacy classes of involutions in $W$ have been classified by Richardson [10, Thm. A] and Springer [11] in terms of the parabolic subgroups of $W$ whose longest element is central. More precisely, each involution is conjugate to a longest element $w_J$ which is central in $W_J$.

The normalisers of parabolic subgroups of finite Coxeter groups have been described by Howlett [7] and Brink and Howlett [2]. Accordingly, the normaliser $N_W(W_J)$ of $W_J$ in $W$ is a semi-direct product of the form $W_J \rtimes N_J$, where $N_J$ is itself a semi-direct product of a Coxeter group of known type and a group $M_J$, [7, Cor. 7]. It turns out, however, that in the case where $w_J$ is central in $W_J$ this group $M_J$ is trivial.

Proposition 2.1. $M_J$ acts faithfully as inner graph automorphisms on $W_J$. In particular, if $w_J$ is central in $W_J$, then $M_J = \{1\}$.

Proof. According to the tables in [2], a non-trivial generator of the group $M_J$ arises from a situation, where $|S \setminus J| = 2$ and either $W$ is of type $E_7$ and $W_J$ is of type $A_4 \times A_1$, or $W$ is of type $D_{2n}$ and $W_J$ is of type $A_{2n-2}$ or $A_{2k} \times A_{2l}$ with $k \neq l$ and $k + l = n - 1$. Let us say that $W_J$ is an $M$-parabolic subgroup of $W$ in such a case. An easy check shows that, if $W_J$ is an $M$-parabolic subgroup of $W$ then $M_J$ induces the same non-trivial graph automorphism on $W_J$ as conjugation by $w_J$. In general, it follows that $M_J$ is trivial unless a conjugate $L$ of $J$ lies in a subset $K \subseteq S$ such that $|K \setminus L| = 2$ and $W_K = W_N \times W_K'$ and $W_L = W_N \times W_{L'}$ for suitable subsets $K', L', N \subseteq S$ and $W_{L'}$ is an $M$-parabolic subgroup of $W_K$. Now $M_L$ induces a non-trivial inner automorphism on $W_L$ and so does $M_J$ on $W_J$.

By [7, Cor. 9], $M_J$ intersects the centraliser of $J$ in $N_J$ trivially, and hence acts faithfully on $W_J$. \hfill \Box

The centraliser of the involution $w_J$ and the normaliser of the parabolic subgroup $W_J$ of $W$ coincide; see [3, Prop. 7]. We give a new proof of this property.

Proposition 2.2. Let $J \subseteq S$ be such that $w_J$ is central in $W_J$. Then $C_W(w_J) = N_W(W_J)$.
Proof. Clearly, $W_J \subseteq C_W(w_J) \cap N_W(W_J)$. It thus suffices to consider the set $D_J = \{ x \in W : l(sx) > l(x) \text{ and } l(xs) > l(x) \text{ for all } s \in J \}$ of distinguished double coset representatives of $W_J$ in $W$.

We have $l(w^x) = l(w)$ for all $w \in W_J$, $x \in N_J = \{ x \in D_J : J^x = J \}$. In particular, $w_J^x = w_J$ for $x \in N_J$. Hence $N_W(W_J) \subseteq C_W(w_J)$.

Conversely, let $x \in C_W(w_J) \cap D_J$. Then $w_J \in W_J \cap W_J^x = W_{J \cap J^x}$; cf. [6, (2.1.12)]. It follows that $J = J^x$ whence $C_W(w_J) \subseteq N_W(W_J)$.

We call the parabolic subgroup $W_J$ bulky (in $W$) if $N_W(W_J) = W_J \times N_J$, i.e., if $N_J$ acts trivially on $W_J$. The main result of this note is the following theorem.

**Theorem 2.3.** Let $J \subseteq S$ be such that $w_J$ is central in $W_J$. Then the involution $w_J$ is special if and only if $W_J$ is bulky.

In our arguments we do make use of the classification of the irreducible Coxeter groups and the structure of the root systems of Weyl groups. Also, we use the notation and labelling of the Dynkin diagram of $W$ as in [1, Planches I - IX].

3. SPECIAL INVOLUTIONS AND BULKY PARABOLIC SUBGROUPS

We maintain the notation from the previous sections.

**Lemma 3.1.** If dim $V_1 = 1$ and $\Phi_1 \neq \emptyset$ or if dim $V_{-1} = 1$, then $w_J$ is special. In particular, $\pm s$ is special for every reflection $s \in T$.

**Proof.** The projection of any root onto a one-dimensional space generated by a root $\alpha$ is clearly proportional to $\alpha$. \hfill $\Box$

**Remark 3.2.** The element $w_J$ is central in $W_J$ if and only if $W_J$ has no components of type $A_n$ with $n \geq 2$, of type $D_{2n+1}$ with $n \geq 2$, of type $E_6$, or of type $I_2(2m+1)$, $m \geq 2$; see [10, 1.12].

**Proof of Theorem 2.3.** We may assume that $W$ is irreducible. By [2, Thm. B] and our Proposition 2.1 the group $N_J$ is generated by certain conjugates of elements of the form $w_L w_K$, where $L \subseteq K \subseteq S$ such that $L$ is a conjugate of $J$, $|K \setminus L| = 1$ and $L^{w_K} = L$. If $s^{w_L w_K} = s$ for all $s \in L$, then $w_L w_K$ centralises $W_L$ and so its conjugate centralises $W_J$. Obviously, $s^{w_L w_K} = s^{w_K}$ for all $s \in L$, since $w_L$ is central in $W_L$.

Now suppose that $W_J$ is not bulky in $W$, i.e., that $N_J$ does not centralise $W_J$. Then there exists a conjugate $L$ of $J$ and a subset $K \subseteq S$ such that $L \subseteq K$ with $|K \setminus L| = 1$ and $w_K$ induces a non-trivial graph automorphism on $W_L$. It follows that $W_K = W_N \times W_{K'}$ for suitable $N, K' \subseteq S$ where the type of $W_K$ is one of those listed in Remark 3.2. Since $w_L$ is central in $W_L$, it follows that $W_L = W_N \times W_{L'}$ where $W_{L'}$ is
span of $\varepsilon$ is proportional to any root in $\Phi$. It is easy to check that $D$ be a root system of type $\Phi$ be the root system of type $A$, $a$ direct product of components of type $A$ and $N$ one such component, $A$ type $N$ permutes them non-trivially. Hence $W_J$ is of type $A_1$ and the claim follows by Lemma 3.1.

If $W$ is of type $A_n$ ($n \geq 1$), then, by Remark 3.2, $W_J$ is necessarily a direct product of components of type $A_1$. But if there is more than one such component, $N_J$ permutes them non-trivially. Hence $W_J$ is of type $A_1$ and the claim follows by Lemma 3.1.

If $W$ is of type $C_n$ ($n \geq 2$), then, by Remark 3.2, $W_J$ is a direct product of a component of type $C_m$, $0 \leq m < n$ and further components of type $A_1$. As before there cannot be more than one component of type $A_1$. Hence $W_J$ is of type $C_m$ or of type $C_m \times A_1$ for some $m < n$. In any case $W_J$ has a component of type $C_m$.

Let $\Phi = \{\pm 2\varepsilon_i : 1 \leq i \leq n\} \cup \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}$ be the root system of type $C_n$. Consider the maximal rank subsystem $\Phi'$ of type $C_m \times C_{n-m}$ consisting of the long roots $\{\pm 2\varepsilon_i : 1 \leq i \leq n\}$ and the short roots $\{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m \text{ or } m + 1 \leq i < j \leq n\}$. Let $U_1$ be the subspace of $V$ spanned by $\varepsilon_1, \ldots, \varepsilon_m$ and $U_2$ the subspace spanned by $\varepsilon_{m+1}, \ldots, \varepsilon_n$. Then $U_2 \cap \Phi$ is a root system of type $C_{n-m}$. All the long roots $\pm 2\varepsilon_i$ of $\Phi$ are contained in $\Phi'$. A short root $\pm \varepsilon_i \pm \varepsilon_j$ is either contained in $\Phi'$ or both its projections on $U_1$ and $U_2$ are proportional to a root in $\Phi'$. By construction, the $-1$-eigenspace $V_{-1}$ of $w_J$ contains $U_1$. Hence every root that lies in
$U_1$ or is proportional to a root in $U_1$ is also proportional to a root in $V_{-1}$. It remains to consider the roots in $U_2$. Without loss we can now assume that $m = 0$. Then $W_J$ is of type $A_1$ and the claim follows by Lemma 3.1.

If $W$ is of type $D_{2n+1}$ ($n \geq 2$), then, by Remark 3.2, $W_J$ is a direct product of an optional component of type $D_{2m}$, $1 \leq m \leq n$ and further components of type $A_1$. As before there cannot be more than one component of type $A_1$. And $N_J$ acts non-trivially on a component of type $D_{2m}$. Hence $W_J$ is of type $A_1$ and the claim follows by Lemma 3.1.

If $W$ is of type $D_{2n}$ ($n \geq 2$), then, by Remark 3.2, $W_J$ is a direct product of an optional component of type $D_{2m}$, $1 \leq m < n$ and further components of type $A_1$. As before there cannot be more than one component of type $A_1$. The non-trivial action of the parabolic subgroup of type $D_{2m+1}$ on a component of type $D_{2m}$ then restricts the type of $W_J$ to either $A_1$ or $D_{2(n-1)} \times A_1$. In the latter case $V_1 \cap \Phi$ is a root system of type $A_1$ and so in both cases the claim follows by Lemma 3.1.

If $W$ is of type $I_2(m)$ ($m \geq 5$), then $W_J$ is of type $A_1$ and the claim follows by Lemma 3.1.

Finally, if $W$ has type $E_6$, $E_7$, $E_8$, $F_4$, $H_3$, or $H_4$, then the claim is established by inspection. \hfill \Box

Remark 3.3. Felder and Veselov prove the one implication of Theorem 2.3, namely that $W_J$ is bulky if $w_J$ is special in a case by case analysis [3, Prop. 10].

Remark 3.4. Bulky parabolic subgroups can be easily classified. It turns out that if $W$ has a central longest element, then $w_J$ is central in $W_J$ whenever $W_J$ is bulky. Otherwise, $W$ has bulky parabolic subgroups $W_J$ which are not associated with a conjugacy class of involutions in $W$.

Acknowledgements: This paper was written while the first author was visiting the School of Mathematics and Statistics of the University of Birmingham under the Scheme 2 LMS grant no. 2925. We are grateful to the LMS for its financial support and to the members of the School for their hospitality. We also wish to thank A. Borovik for bringing the problem of an intrinsic characterisation of special involutions to our attention.

References


