AROUND SOLOMON'S DESCENT ALGEBRAS

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Abstract. We study different problems related to the Solomon's descent algebra $\Sigma(W)$ of a finite Coxeter group (W,S): positive elements, morphisms between descent algebras, Loewy length... One of the main result is that, if W is irreducible and if the longest element is central, then the Loewy length of $\Sigma(W)$ is equal to $\left\lceil \frac{|S|}{2} \right\rceil$.

Introduction

Let (W,S) be a finite Coxeter system. The descent algebra $\Sigma(W)$ of the finite Coxeter group W is a subalgebra of the group algebra $\mathbb{Q}W$ with a basis $\{x_I : I \subset S\}$, where x_I is the sum in $\mathbb{Q}W$ of the distinguished coset representatives of the parabolic subgroup W_I in W. It is a non-commutative preimage of the ring of parabolic permutation characters of W, with respect to the homomorphism θ which associates to x_I the permutation character of W on the cosets of W_I. Solomon [S] discovered it as the real reason why the sign character of W is a linear combination of parabolic permutation characters. He also showed that $\operatorname{Ker}\theta$ is the radical of $\Sigma(W)$.

The special case where W is the symmetric group on n points, i.e., a Coxeter group of type A_{n-1}, has received particular attention. This type of descent algebra occurs as the dual of the Hopf algebra of quasi-symmetric functions. Atkinson [A] has determined the Loewy length of $\Sigma(W)$ in this case.

For general W, the descent algebra has been further studied as an interesting object in its own right. Bergeron, Bergeron, Howlett and Taylor [BBHT] have constructed explicit idempotents, decomposing $\Sigma(W)$ into projective indecomposable modules. Recently, Blessenohl, Hohlweg and Schocker [BHS] could show that θ satisfies the remarkable symmetry $\theta(x)(y) = \theta(y)(x)$ for all $x, y \in \Sigma(W)$.

The main purpose of this article is to determine the Loewy length of $\Sigma(W)$ for all types of irreducible finite Coxeter groups W. With the exception of type D_n, n odd, this is done through a case by case analysis, using computer calculations with CHEVIE [Chevie] for the exceptional types, in the final Section 5. Our results show in particular, that if W is irreducible and if the longest element w_0 is central in W then the Loewy length of $\Sigma(W)$ is exactly $\left\lceil \frac{|S|}{2} \right\rceil$, whereas in the other cases, it lies between $\left\lceil \frac{|S|}{2} \right\rceil$ and $|S|$. Moreover, in Section 3, we study ideals generated by elements of $\Sigma^+(W)$, the set of non-negative linear combinations of the basis elements x_I of $\Sigma(W)$, and show that the minimal polynomial of an element of $\Sigma^+(W)$ is square-free. Section 4 deals with various types of homomorphisms between descent algebras, some restriction morphisms and one type related to self-opposed subsets. A restriction morphism between the descent algebra of type B_n and the descent algebra of type D_n is also defined. Section 2 sets the scene in terms of a finite

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Coxeter group W and a length-preserving automorphism σ. The general object of interest is $\Sigma(W)^\sigma$, the subalgebra of fixed points of σ in $\Sigma(W)$.

REMARK - If W_n is a Weyl group of type B_n, there exists an extension Σ'_n of the descent algebra $\Sigma(W_n)$ which was defined by Mantaci and Reutenauer [MR] and studied by Hohlweg and the first author [BH]. In [B], the first author investigates similar problems for this algebra (restriction morphisms, positive elements, Loewy series...): for instance, Σ'_n has Loewy length n.

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1. NOTATION, PRELIMINARIES

1.A. General notation. If X is a set, $\mathcal{P}(X)$ denotes the set of subsets of X and $\mathcal{P}^{\#}(X)$ denotes the set of proper subsets of X. If $k \in \mathbb{Z}$, we denote by $\mathcal{P}_{\leqslant k}(X)$ the set of subsets I of X such that $|I| \leqslant k$. The group algebra of a group G over $\mathbb{Q}$ is denoted by $\mathbb{Q}G$. If G is a finite group, let $\operatorname{Irr} G$ denote the set of its (ordinary) irreducible characters over $\mathbb{C}$. The Grothendieck group of the category of finite dimensional $\mathbb{C}G$-modules is identified naturally with the free $\mathbb{Z}$-module $\mathbb{Z}\operatorname{Irr} G$ and we set $\mathbb{Q}\operatorname{Irr} G = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}\operatorname{Irr} G$. If A is a finite dimensional $\mathbb{Q}$-algebra, we denote by $\operatorname{Rad} A$ its radical. If $a \in A$, the centralizer of a in A is denoted by $Z_A(a)$. The set of irreducible characters of A is denoted by $\operatorname{Irr} A$.

1.B. Coxeter groups. Let (W, S) be a finite Coxeter group. Let $\ell : W \to \mathbb{N}$ be the length function attached to S and let $\leqslant$ denote the Bruhat-Chevalley order on W. Let w_0 denote the longest element of W. If $I \in \mathcal{P}(S)$, let W_I denote the subgroup of W generated by I. Recall that (W_I, I) is a Coxeter group. The trivial character of W_I is denoted by 1_I. A *parabolic subgroup* of W is a subgroup of W which is conjugate to some W_I.

1.C. Solomon descent algebra. If $I \subset S$, we set

$$X_I = \{w \in W \mid \forall\ s \in I,\ ws > w\}.$$

Recall that an element $w \in W$ lies in X_I if and only if $w(\Delta_I) \subset \Phi^+$. Let

$$x_I = \sum_{w \in X_I} w \in \mathbb{Q}W.$$

Let

$$\Sigma(W) = \mathop{\oplus}_{I \in \mathcal{P}(S)} \mathbb{Q}x_I \quad \subset \quad \mathbb{Q}W.$$

If $\mathcal{F}$ is a subset of $\mathcal{P}(S)$, we set

$$\Sigma_{\mathcal{F}}(W) = \mathop{\oplus}_{I \in \mathcal{F}} \mathbb{Q}x_I.$$

In particular, $\Sigma_{\mathcal{P}(S)}(W) = \Sigma(W)$. Let $\theta : \Sigma(W) \to \mathbb{Q}\operatorname{Irr} W$ be the unique linear map such that $\theta(x_I) = \operatorname{Ind}_{W_I}^W 1_I$ for every $I \subset S$. Let $(\xi_I)_{I \in \mathcal{P}(S)}$ denote the $\mathbb{Q}$-basis of $\operatorname{Hom}_{\mathbb{Q}}(\Sigma(W), \mathbb{Q})$ dual to $(x_I)_{I \in \mathcal{P}(S)}$. In other words,

$$x = \sum_{I \in \mathcal{P}(S)} \xi_I(x) x_I$$

for every $x \in \Sigma(W)$. If $s \in S$, we write x_s (resp. ξ_s) for $x_{\{s\}}$ (resp. $\xi_{\{s\}}$) for simplification.

If I and J are two subsets of S, we set

$$X_{IJ} = (X_I)^{-1} \cap X_J.$$

We write $I \equiv J$ if there exists $w \in W$ such that $J = {}^w I$ (or, equivalently, if W_I and W_J are conjugate subgroups of W). The relation $\equiv$ is an equivalence relation on $\mathcal{P}(S)$ and we denote by Λ the set of equivalence classes for this relation: it parametrizes the W-conjugacy classes of parabolic subgroups of W. We still denote by $\subset$ the order relation on Λ induced by inclusion. Let $\boldsymbol{\lambda} : \mathcal{P}(S) \to \Lambda$ be the canonical surjection. We can now recall the following result of Solomon [S].

Solomon's Theorem. *With the previous notation, we have:*

(a) *If I and J are two subsets of S, then*

$$x_I x_J = \sum_{d \in X_{IJ}} x_{{}^{d^{-1}}I \cap J}.$$

(b) *$\Sigma(W)$ is a unitary sub-$\mathbb{Q}$-algebra of $\mathbb{Q}W$.*

(c) *$\theta : \Sigma(W) \to \mathbb{Q}\,\mathrm{Irr}\,W$ is a morphism of $\mathbb{Q}$-algebras.*

(d) $\mathrm{Ker}\,\theta = \displaystyle\sum_{I \equiv J} \mathbb{Q}(x_I - x_J)$.

(e) $\mathrm{Rad}\,\Sigma(W) = \mathrm{Ker}\,\theta$.

$\Sigma(W)$ is called *Solomon's descent algebra* of W. If I, J and K are three subsets of S, we set

$$X_{IJK} = \{d \in X_{IJ} \mid {}^{d^{-1}}I \cap J = K\}.$$

Then, Solomon's Theorem (a) can be restated as follows:

(1.1)
$$x_I x_J = \sum_{K \in \mathcal{P}(S)} |X_{IJK}|\, x_K.$$

1.D. Simple $\boldsymbol{\Sigma(W)}$-modules. The intersection of two parabolic subgroups of W is a parabolic subgroup. Therefore, if $w \in W$, we define $W(w)$ to be the minimal parabolic subgroup of W containing w. We denote by $\boldsymbol{\Lambda}(w) \in \Lambda$ the parameter of its conjugacy class. The map

$$\boldsymbol{\Lambda} : W \longrightarrow \Lambda$$

is constant on conjugacy classes and is surjective: indeed, if $\lambda \in \Lambda$, if $I \in \lambda$, and if c is a Coxeter element of W_I, then $\boldsymbol{\Lambda}(c) = \lambda$. The inverse image of $\lambda \in \Lambda$ in W is denoted by $\mathcal{C}(\lambda)$. It is a union of conjugacy classes of W.

If $\lambda \in \Lambda$, let $\tau_\lambda : \Sigma(W) \to \mathbb{Q}$, $x \mapsto \theta(x)(w)$, where $w \in \mathcal{C}(\lambda)$. Recall that $\theta(x)$ is a $\mathbb{Q}$-linear combination of permutation characters, so $\theta(x)(w)$ lies in $\mathbb{Q}$. Moreover, τ_λ does not depend on the choice of w in $\mathcal{C}(\lambda)$, and is a morphism of algebras. Also, the map

$$\begin{array}{rccc} \tau : & \Lambda & \longrightarrow & \mathrm{Irr}\,\Sigma(W) \\ & \lambda & \longmapsto & \tau_\lambda \end{array}$$

is bijective. By definition, if $w \in W$ and $x \in \Sigma(W)$, then

(1.2)
$$\tau_{\boldsymbol{\Lambda}(w)}(x) = \theta(x)(w).$$

Finally, recall that

(1.3)
$$\tau_{\boldsymbol{\lambda}(J)}(x_I) = |X_{IJJ}|.$$

It follows that

(1.4)
$$x x_J - \tau_{\boldsymbol{\lambda}(J)}(x) x_J \in \Sigma_{\mathcal{P}^\#(J)}(W)$$

for every $x \in \Sigma(W)$.

2. Automorphisms of Coxeter groups

2.A. General case. We fix in this section an automorphism σ of W such that $\sigma(S) = S$. Since $\ell \circ \sigma = \ell$, σ induces an automorphism of $\Sigma(W)$ which is still denoted by σ. The subalgebra of fixed points of σ in $\Sigma(W)$ is denoted by $\Sigma(W)^\sigma$.

Lemma 2.1. *Let A be a sub-$\mathbb{Q}$-algebra of $\Sigma(W)$. Then $\operatorname{Rad} A = A \cap \operatorname{Rad}\Sigma(W)$.*

Proof. Let $I = A \cap \operatorname{Rad}\Sigma(W)$. Since $\Sigma(W)$ is basic (i.e., all its simple modules are of dimension 1), $\operatorname{Rad}\Sigma(W)$ is exactly the set of nilpotent elements of $\Sigma(W)$. Therefore, $\operatorname{Rad} A \subset \operatorname{Rad}\Sigma(W)$. In particular, $\operatorname{Rad} A \subset I$. Moreover, I is a two-sided nilpotent ideal of A. So $I \subset \operatorname{Rad} A$ and we are done. $\square$

Corollary 2.2. $\operatorname{Rad}\big(\Sigma(W)^\sigma\big) = \big(\operatorname{Rad}\Sigma(W)\big)^\sigma$.

The automorphism σ acts on $\mathcal{P}(S)$ and this action induces an action of σ on Λ. The set of σ-orbits in Λ is denoted by Λ/σ. It is easily checked that

(2.3) $$\tau_\lambda \circ \sigma^{-1} = \tau_{\sigma(\lambda)}$$

for every $\lambda \in \Lambda$. In particular, if we denote by τ_λ^σ the restriction of τ_λ to $\Sigma(W)^\sigma$, then

(2.4) $$\tau_\lambda^\sigma = \tau_{\sigma(\lambda)}^\sigma.$$

It is also clear that τ_λ^σ is an irreducible character of $\Sigma(W)^\sigma$.

Proposition 2.5. *The map $\Lambda \to \operatorname{Irr}\big(\Sigma(W)^\sigma\big)$, $\lambda \mapsto \tau_\lambda^\sigma$ induces a bijection $\Lambda/\sigma \simeq \operatorname{Irr}\big(\Sigma(W)^\sigma\big)$.*

Proof. By Corollary 2.2 and since $\mathbb{Q}$ has characteristic 0, θ induces an isomorphism of algebras
$$\Sigma(W)^\sigma / \operatorname{Rad}\big(\Sigma(W)^\sigma\big) \simeq (\operatorname{Im}\theta)^\sigma.$$
So we have a natural bijection between $\operatorname{Irr}\Sigma(W)^\sigma$ and $\operatorname{Irr}(\operatorname{Im}\theta)^\sigma$. If $\lambda \in \Lambda$, let e_λ be the idempotent of $\operatorname{Im}\theta$ such that $(\operatorname{Im}\theta)e_\lambda$ is a simple $\Sigma(W)$-module affording τ_λ. Then
$$\operatorname{Im}\theta = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}e_\lambda$$
and $\sigma(e_\lambda) = e_{\sigma(\lambda)}$. So,
$$(\operatorname{Im}\theta)^\sigma = \bigoplus_{\Omega\in\Lambda/\sigma} \mathbb{Q}\Big(\sum_{\lambda\in\Omega} e_\lambda\Big).$$
This completes the proof of the proposition. $\square$

2.B. Action of w_0. Let σ_0 denote the automorphism of W induced by conjugation by w_0, the longest element of W. Then $\sigma_0(S) = S$, so σ_0 induces an automorphism of $\Sigma(W)$. Of course, we have

(2.6) $$\Sigma(W)^{\sigma_0} = Z_{\Sigma(W)}(w_0).$$

Let us introduce another classical basis of $\Sigma(W)$. If $w \in W$, we set
$$\mathcal{R}(w) = \{s \in S \mid ws > w\}.$$
Then

(2.7) $$\mathcal{R}(w_0 w) = S \setminus \mathcal{R}(w).$$

If $J \in \mathcal{P}(S)$, we set
$$Y_J = \{w \in W \mid \mathcal{R}(w) = J\}$$
and
$$y_J = \sum_{w\in Y_J} w \in \mathbb{Q}W.$$

Then

(2.8)
$$x_I = \sum_{I\subset J} y_J,$$

so $y_J \in \Sigma(W)$ and $(y_J)_{J\in\mathcal{P}(S)}$ is a $\mathbb{Q}$-basis of $\Sigma(W)$. Note that $y_S = \{1\}$ and $y_\varnothing = \{w_0\}$, so $w_0 \in \Sigma(W)$. By 2.7, we have

(2.9)
$$y_\varnothing y_J = w_0 y_J = y_{S\setminus J}.$$

The centrality of w_0 can be characterized by the invertibilty of the elements y_J.

Proposition 2.10. *The longest element w_0 is central in W if and only if y_J is invertible for all $J \in \mathcal{P}(S)$.*

Proof. Clearly $x \in \Sigma(W)$ is invertible if and only if $0 \notin \{\theta(x)(w) : w \in W\}$. Moreover, $w_0 \notin W_I$ unless $I = S$. And by Möbius inversion,

$$y_J = \sum_{I\supset J} (-1)^{|I|-|J|} x_I = x_S + \sum_{I\in\mathcal{P}^\#(S):J\subset I} (-1)^{|I|-|J|} x_I.$$

Suppose w_0 is central in W. Then $w_0 \in N_W(W_I)$ for all $I \in \mathcal{P}(S)$ and the index $|N_W(W_I) : W_I|$ is even for $I \in \mathcal{P}^\#(S)$. Let $w \in W$. Then $\theta(x_I)(w)$, which is a multiple of $|N_W(W_I) : W_I|$, is even for $I \in \mathcal{P}^\#(S)$. And $\theta(y_J)(w)$, which is the sum of $\pm\theta(x_I)(w)$ for certain $I \in \mathcal{P}^\#(S)$ and $\theta(x_S)(w) = 1$ is odd, in particular not zero.

Conversely, if w_0 is not central in W, there is a maximal proper subset $I \subset S$ such that $I^{w_0} \neq I$. (Otherwise $s^{w_0} = s$ for all $s \in S$, in contradiction to w_0 being non-central.) It follows that, if w is an element of the same shape as I, then $\theta(x_I)(w) = |N_W(W_I) : W_I| = 1$. Hence $y_I = x_I - x_S$ implies $\theta(y_I)(w) = 1 - 1 = 0$. $\square$

If $I \in \mathcal{P}(S)$, we set

$$x'_I = \sum_{K\in\mathcal{P}(I)} \Big(-\frac{1}{2}\Big)^{|I|-|K|} x_K.$$

Note that $(x'_I)_{I\in\mathcal{P}(S)}$ is a basis of $\Sigma(W)$. Using 2.8, it is easily checked that

(2.11)
$$x'_I = \Big(-\frac{1}{2}\Big)^{|I|} \sum_{J\in\mathcal{P}(S)} (-1)^{|I\cap J|} y_J.$$

Therefore, by 2.9, we get

(2.12)
$$w_0 x'_I = (-1)^{|I|} x'_I.$$

So, if w_0 is central in W, we can improve 1.4:

Lemma 2.13. *Let $I \in \mathcal{P}(S)$ and $x \in \Sigma(W)^{\sigma_0}$. Then*

$$x x'_I \in \tau_{\boldsymbol{\lambda}(I)}(x) x'_I + \Sigma_{\mathcal{P}_{\leqslant\, |I|-2}(I)}(W).$$

Proof. Let us write

$$x x'_I = \sum_{J\subset I} \alpha_J x'_J.$$

Evaluating ξ_I on each side, we get that $\alpha_I = \tau_{\boldsymbol{\lambda}(I)}(x)$ (see 1.4). Since x commutes with w_0, it follows from 2.12 that $\alpha_J = 0$ if $|J| - |I| \equiv 1 \mod 2$, as desired. $\square$

3. Positivity properties

We denote by $\Sigma^+(W)$ the set of elements $a \in \Sigma(W)$ such that $\xi_I(a) \geqslant 0$ for every $I \in \mathcal{P}(S)$. Note that $x_I \in \Sigma^+(W)$ for every $I \in \mathcal{P}(S)$. If $a, b \in \Sigma^+(W)$, then

(3.1) $$a + b \in \Sigma^+(W)$$

and, by Solomon's Theorem (a),

(3.2) $$ab \in \Sigma^+(W).$$

The aim of this section is to study properties of the elements of $\Sigma^+(W)$ (ideals generated, minimal polynomial, centralizer...).

3.A. Ideals. A subset $\mathcal{F}$ of $\mathcal{P}(S)$ is called *saturated* (resp. *equivariantly saturated*) if, for every $I \in \mathcal{F}$ and every $I' \in \mathcal{P}(S)$ such that $I' \subset I$ (resp. $\boldsymbol{\lambda}(I') \subset \boldsymbol{\lambda}(I)$), we have $I' \in \mathcal{F}$. If $\mathcal{F}$ is equivariantly saturated, then it is saturated. If $\mathcal{F}$ is saturated (resp. equivariantly saturated) then, by Solomon's Theorem (a), $\Sigma_{\mathcal{F}}(W)$ is a left (resp. two-sided) ideal of $\Sigma(W)$.

Example and notation - Then $\mathcal{P}_{\leqslant k}(S)$ is an equivariantly saturated subset of $\mathcal{P}(S)$. Moreover, if $I \subset S$, then $\mathcal{P}(I)$ and $\mathcal{P}^{\#}(I)$ are saturated subsets of $\mathcal{P}(S)$. □

Proposition 3.3. *Let $\mathcal{F}$ be a saturated subset of $\mathcal{P}(S)$ and let $\chi_{\mathcal{F}}$ denote the character of the left $\Sigma(W)$-module $\Sigma_{\mathcal{F}}(W)$. Then*

$$\chi_{\mathcal{F}} = \sum_{I \in \mathcal{F}} \tau_{\boldsymbol{\lambda}(I)}.$$

Proof. This follows immediately from 1.4. □

If $a \in \Sigma(W)$, we set

$$\mathcal{F}(a) = \{I \in \mathcal{P}(S) \mid \exists\ J \in \mathcal{P}(S),\ \big(\xi_J(a) \neq 0 \text{ and } I \subset J\big)\}$$

$$\mathcal{F}_{\mathrm{eq}}(a)\{I \in \mathcal{P}(S) \mid \exists\ J \in \mathcal{P}(S),\ \big(\xi_J(a) \neq 0 \text{ and } \boldsymbol{\lambda}(I) \subset \boldsymbol{\lambda}(J)\big)\}.$$

Note that $\mathcal{F}(a) \subset \mathcal{F}_{\mathrm{eq}}(a)$. Then $\mathcal{F}(a)$ (resp. $\mathcal{F}_{\mathrm{eq}}(a)$) is saturated (resp. equivariantly saturated) and, by Solomon's Theorem (a),

(3.4) $$\Sigma(W)a \subset \Sigma_{\mathcal{F}(a)}(W)$$

and

(3.5) $$a\Sigma(W) \subset \Sigma_{\mathcal{F}_{\mathrm{eq}}(a)}(W).$$

The next proposition shows that equality holds in 3.5 whenever $a \in \Sigma^+(W)$.

Proposition 3.6. *Let $a \in \Sigma^+(W)$. Then*

$$a\Sigma(W) = \Sigma_{\mathcal{F}_{\mathrm{eq}}(a)}(W).$$

In particular, $\Sigma(W)a \subset a\Sigma(W)$.

Proof. We may, and we will, assume that $a \neq 0$. Let $\mathcal{F} = \mathcal{F}_{\mathrm{eq}}(a)$ and $\mathcal{I} = a\Sigma(W)$. Then $\mathcal{F}$ is equivariantly saturated and $\mathcal{I} \subset \Sigma_{\mathcal{F}}(W)$ (see 3.5). Now let $I \in \mathcal{F}$. We shall show by induction on $|I|$ that $x_I \in \mathcal{I}$.

First, note that

$$ax_\varnothing = \Big(\sum_{I \in \mathcal{P}(S)} |X_I|\xi_I(a)\Big)x_\varnothing$$

so, by hypothesis, $ax_\varnothing = mx_\varnothing$ with $m > 0$. Therefore, $x_\varnothing \in \mathcal{I}$. Now, let $I \in \mathcal{F}$ and assume that, for every $J \in \mathcal{F}$ such that $|J| \leqslant |I| - 1$, we have $x_J \in \mathcal{I}$. We want to prove that $x_I \in \mathcal{I}$. Let $I_0 \in \mathcal{P}(S)$ be such that $\boldsymbol{\lambda}(I) \subset \boldsymbol{\lambda}(I_0)$ and $\xi_{I_0}(a) \neq 0$. By the positivity of a and by Solomon's Theorem (a), this shows that $\xi_I(ax_I) > 0$.

But $ax_I = \sum_{J \in \mathcal{P}(I)} \xi_J(ax_I)x_J$. Since $ax_I \in \mathcal{I}$ and $x_J \in \mathcal{I}$ for every $J \in \mathcal{P}^{\#}(I)$, we get that $x_I \in \mathcal{I}$, as desired. □

Corollary 3.7. *Let $a \in \Sigma^+(W)$. Then a is invertible in $\Sigma(W)$ if and only if $\xi_S(a) > 0$.*

Corollary 3.8. *Let $a_1,\dots, a_r \in \Sigma^+(W)$. Then $a_1 + \cdots + a_r \in \Sigma^+(W)$ and*

$$a_1\Sigma(W) + \cdots + a_r\Sigma(W) = (a_1 + \cdots + a_r)\Sigma(W).$$

Proof. By Proposition 3.6, we have

$$a_1\Sigma(W) + \cdots + a_r\Sigma(W) = \Sigma_{\mathcal{F}_{\mathrm{eq}}(a_1)\cup\cdots\cup\mathcal{F}_{\mathrm{eq}}(a_r)}(W).$$

But it is clear that

$$\mathcal{F}_{\mathrm{eq}}(a_1) \cup \cdots \cup \mathcal{F}_{\mathrm{eq}}(a_r) = \mathcal{F}_{\mathrm{eq}}(a_1 + \cdots + a_r).$$

By applying Proposition 3.6 to $a = a_1 + \cdots + a_r$, we get the desired result. □

3.B. Minimal polynomial. If $a \in \Sigma(W)$, we denote by $f_a(T) \in \mathbb{Q}[T]$ its minimal polynomial. Let $m_a : \Sigma(W) \to \Sigma(W)$, $x \mapsto ax$ be the left multiplication by a and let M_a be the matrix of m_a in the basis $(x_J)_{J\in\mathcal{P}(S)}$. The minimal polynomial of a is equal to the minimal polynomial of the linear map m_a (or of the matrix M_a). By 1.4, M_a is triangular (with respect to the order $\subset$ on $\mathcal{P}(S)$) and its characteristic polynomial is

$$\prod_{J\in\mathcal{P}(S)} (T - \tau_{\boldsymbol{\lambda}(J)}(a)).$$

In particular

(3.9) $\qquad$ *f_a is split over $\mathbb{Q}$.*

The main result of this subsection is the following:

Proposition 3.10. *Let $a \in \Sigma^+(W)$. Then f_a is square-free.*

Proof. Before starting the proof, we gather in the next lemma some elementary properties of elements of $\Sigma^+(W)$.

Lemma 3.11. *Let I, J and K be three subsets of S such that $J \subset K$ and let $a \in \Sigma^+(W)$. Then:*
(a) *$X_{IK} \subset X_{IJ}$ and $X_{IKK} \subset X_{IJJ}$.*
(b) *$\tau_{\boldsymbol{\lambda}(K)}(a) \leqslant \tau_{\boldsymbol{\lambda}(J)}(a)$.*
(c) *If $\tau_{\boldsymbol{\lambda}(K)}(a) = \tau_{\boldsymbol{\lambda}(J)}(a)$ and if $\xi_I(a) \neq 0$, then:*
  (c1) *$X_{IJJ} = X_{IKK}$.*
  (c2) *If $J \subsetneq K$, then $X_{IKJ} = \varnothing$.*

Proof of Lemma 3.11. It is clear that $X_{IK} \subset X_{IJ}$. Now, we have $X_{IKK} = \{d \in X_{IK} \mid K \subset {}^{d^{-1}}I\}$, so (a) follows. Now, by 1.3, we have

$$\tau_{\boldsymbol{\lambda}(K)}(a) = \sum_{I\in\mathcal{P}(S)} \xi_I(a)|X_{IKK}|.$$

So (b) and (c1) follow immediately from (a) and this equality. Let us now prove (c2). So assume that $\tau_{\boldsymbol{\lambda}(K)}(a) = \tau_{\boldsymbol{\lambda}(J)}(a)$, that $\xi_I(a) \neq 0$ and that $X_{IKJ} \neq \varnothing$. Let $d \in X_{IKJ}$. Then $d \in X_{IJ}$ by (a) and $J = {}^{d^{-1}}I \cap K \subset {}^{d^{-1}}I$. In other words, $d \in X_{IJJ}$. Therefore, $d \in X_{IKK}$ by (c1) and, since $J = {}^{d^{-1}}I \cap K$, we have $J = K$, as expected. □

Let $\xi \in \mathbb{Q}$ be an eigenvalue of m_a. Let $\mathcal{F} = \{J \in \mathcal{P}(S) \mid \tau_{\boldsymbol{\lambda}(J)}(a) = \xi\}$. Note that $\mathcal{F} \neq \varnothing$. Since the matrix $M_a = (\xi_J(ax_K))_{K,J\in\mathcal{P}(S)}$ is triangular, it is sufficient to show that the square matrix $(\xi_J(ax_K))_{K,J\in\mathcal{F}}$ is diagonal. So, let J and K be two elements of $\mathcal{F}$ such that $\xi_J(ax_K) \neq 0$. We want to show that $J = K$. First, since $\xi_J(ax_K) \neq 0$, we have $J \subset K$. Moreover, there exists $I \in \mathcal{P}(S)$ such that $\xi_I(a) \neq 0$ and $X_{IKJ} \neq \varnothing$. But, by (2), we have $X_{IJJ} = X_{IKK}$. Now, let $d \in X_{IKJ}$ (such a d exists by hypothesis). Then $d \in X_{IJ}$ and $J = {}^{d^{-1}}I \cap K \subset {}^{d^{-1}}I$. In other words, $d \in X_{IJJ}$. Therefore, $d \in X_{IKK}$ and, since $J = {}^{d^{-1}}I \cap K$, we have $J = K$, as expected. $\square$

Corollary 3.12. *Let $a \in \Sigma^+(W)$ and let $n \geqslant 1$. Then $a^n\Sigma(W) = a\Sigma(W)$ and $\Sigma(W)a^n = \Sigma(W)a$.*

Proof. It is sufficient to prove this result for $n = 2$. If a is invertible, then the result is clear. If a is not invertible then, by Proposition 3.10, the minimal polynomial f_a of a is divisible by T and not by T^2. This shows that $a \in \mathbb{Q}[a]a^2 = \mathbb{Q}[a]a^2$. So $a^2 \in \Sigma(W)a$ and $a^2 \in a\Sigma(W)$, as expected. $\square$

Corollary 3.13. *Let M be a $\Sigma(W)$-module and let χ_M denote its character. Write $\chi_M = \tau_{\lambda_1} + \cdots + \tau_{\lambda_r}$, with $\lambda_1,\ldots,\ \lambda_r \in \Lambda$ (possibly non-distinct). Let $a \in \Sigma^+(W)$ and let $\xi \in \mathbb{Q}$. Then*

$$\dim_{\mathbb{Q}} \operatorname{Ker}(a - \xi \operatorname{Id}_M \mid M) = |\{1 \leqslant i \leqslant r \mid \tau_{\lambda_i}(a) = \xi\}|.$$

Proof. Indeed, if $x \in \Sigma(W)$, then $(\tau_{\lambda_i}(x))_{1 \,\leqslant\, i \,\leqslant\, r}$ is the multiset of eigenvalues of x in its action on M. But, by Proposition 3.10, a acts semisimply on M. This proves the result. $\square$

EXAMPLE 3.14 - Consider here the left $\Sigma(W)$-module $\mathbb{Q}W$, with the natural action by left multiplication. Let χ denote its character. Then it is easy and well-known that

$$\chi(x_I) = |W|$$

for every $I \in \mathcal{P}(S)$. Therefore,

$$\text{(a)} \qquad \chi = \sum_{\lambda\in\Lambda} |\mathcal{C}(\lambda)|\ \tau_\lambda = \sum_{w\in W} \tau_{\boldsymbol{\Lambda}(w)}.$$

Indeed, by 1.2, we have

$$\sum_{w\in W} \tau_{\boldsymbol{\Lambda}(w)}(x_I) = \sum_{w\in W} \theta(x_I)(w) = |W|\langle\theta(x_I), 1_S\rangle = |W|.$$

Therefore, if $a \in \Sigma^+(W)$ and $\xi \in \mathbb{Q}$, it follows from Corollary 3.13 and 1.2 that

$$\text{(b)} \qquad \dim_{\mathbb{Q}} \operatorname{Ker}(a - \xi \operatorname{Id}_{\mathbb{Q}W} \mid \mathbb{Q}W) = |\{w \in W \mid \theta(a)(w) = \xi\}|. \ \square$$

3.C. Centralizers. The aim of this subsection is to prove a few results on the dimension of the centralizer of elements of $\Sigma^+(W)$. We first start with some easy observation.

Let $a \in \Sigma(W)$. Let $\mu_a : \Sigma(W) \to \Sigma(W)$, $x \mapsto ax - xa$. Then

$$\textbf{(3.15)} \qquad \operatorname{Ker} \mu_a = Z_{\Sigma(W)}(a)$$

so that

$$\textbf{(3.16)} \qquad \dim_{\mathbb{Q}} \Sigma(W) - \dim_{\mathbb{Q}} Z_{\Sigma(W)}(a) = \dim_{\mathbb{Q}}(\operatorname{Im} \mu_a).$$

Proposition 3.17. *Let $a \in \Sigma^+(W)$. Then*

$$\dim_{\mathbb{Q}} Z_{\Sigma(W)}(a) = 2^{|S|} - |\mathcal{F}_{\text{eq}}(a)| + \dim_{\mathbb{Q}} \Sigma(W)a - \dim_{\mathbb{Q}}(\operatorname{Im}\mu_a \cap \Sigma(W)a).$$

In particular,

$$\dim_{\mathbb{Q}} Z_{\Sigma(W)}(a) \leqslant 2^{|S|} - |\mathcal{F}_{\text{eq}}(a)| + \dim_{\mathbb{Q}} \Sigma(W)a \leqslant 2^{|S|} - |\mathcal{F}_{\text{eq}}(a)| + |\mathcal{F}(a)|.$$

REMARK - Recall that $\mathcal{F}(a) \subset \mathcal{F}_{\text{eq}}(a)$ so that the right-hand side of the above inequality is always $\leqslant 2^{|S|}$. $\square$

Proof. Let $\pi : a\Sigma(W) \to a\Sigma(W)/\Sigma(W)a$ be the canonical projection. Let $f : \Sigma(W) \to a\Sigma(W)$, $x \mapsto ax$. Then $\pi \circ f$ is surjective by definition. Moreover, note that the image of μ_a is contained in $a\Sigma(W)$. By definition, $\pi\circ f = \pi\circ\mu_a$. Therefore, $\pi \circ \mu_a$ is surjective. In particular,

$$\dim_{\mathbb{Q}}(\operatorname{Im}\mu_a) = |\mathcal{F}_{\text{eq}}(a)| - \dim_{\mathbb{Q}} \Sigma(W)a + \dim_{\mathbb{Q}}(\operatorname{Im}\mu_a \cap \Sigma(W)a).$$

So the result now follows from 3.16. $\square$

EXAMPLE 3.18 - The following example shows that the first inequality in Proposition 3.17 might be strict. Assume here that $W = \mathfrak{S}_4$ is of type A_3. Write $S = \{s_1, s_2, s_3\}$, with $s_1 s_3 = s_3 s_1$. Then

$$\dim_{\mathbb{Q}} Z_{\Sigma(W)}(x_{\{s_1,s_2\}}) = 5$$

and $$2^{|S|} - |\mathcal{F}_{\text{eq}}(x_{\{s_1,s_2\}})| + \dim_{\mathbb{Q}} \Sigma(W)x_{\{s_1,s_2\}} = 6. \ \square$$

Corollary 3.19. *Let $a \in \Sigma^+(W)$ and assume that* $\operatorname{Rad}\Sigma(W) \cap \Sigma(W)a = 0$. *Then*

$$\dim_{\mathbb{Q}} Z_{\Sigma(W)}(a) = 2^{|S|} - |\mathcal{F}_{\text{eq}}(a)| + \dim_{\mathbb{Q}} \Sigma(W)a.$$

Proof. Since $\operatorname{Im}\mu_a \subset \operatorname{Rad}\Sigma(W)$, the hypothesis implies that $\operatorname{Im}\mu_a \cap \Sigma(W)a = 0$. So the result follows now from Proposition 3.17. $\square$

EXAMPLE 3.20 - Let $s \in S$. Let $C(s)$ denote the set of elements of S which are conjugate to s in W and let $c(s) = |C(s)|$. Let $a = \sum_{t\in C(s)} \alpha_t x_t \neq 0$ be such that $\alpha_t \geqslant 0$ for every $t \in C(s)$. Then

(1) $$\mathcal{F}_{\text{eq}}(a) = C(s) \cup \{\varnothing\}.$$

Moreover, if $t \in C(s)$, then $\xi_s(xx_s) = \xi_t(xx_t)$ for every $x \in \Sigma(W)$ (see 1.3 and 1.4). Therefore,

(2) $$\Sigma(W)a = \mathbb{Q}a \oplus \mathbb{Q}x_{\varnothing}.$$

In particular,

(3) $$\operatorname{Rad}\Sigma(W) \cap \Sigma(W)a = 0.$$

It then follows from (1), (2), (3) and Corollary 3.19 that

(4) $$\dim_{\mathbb{Q}} Z_{\Sigma(W)}(a) = 2^{|S|} - c(s) + 1.$$

Note that this equality holds if $a = x_s$. $\square$

Corollary 3.21. *Let $a \in \Sigma^+(W)$ and let $n \geqslant 1$. Then $Z_{\Sigma(W)}(a^n) = Z_{\Sigma(W)}(a)$.*

Proof. Since $Z_{\Sigma(W)}(a^n) \subset Z_{\Sigma(W)}(a)$, we only need to prove that the dimensions of both centralizers are equal. First, by Corollary 3.12, we have $\dim_{\mathbb{Q}} \Sigma(W)a = \dim_{\mathbb{Q}} \Sigma(W)a^n$ and $\dim_{\mathbb{Q}} a\Sigma(W) = \dim_{\mathbb{Q}} a^n\Sigma(W)$. So, by Proposition 3.17, we only need to prove that

(P_n) $$\dim_{\mathbb{Q}}(\operatorname{Im}\mu_{a^n} \cap \Sigma(W)a) = \dim_{\mathbb{Q}}(\operatorname{Im}\mu_a \cap \Sigma(W)a).$$

We will show (P_n) by induction on n, the case where $n = 1$ being trivial. So we assume that $n \geqslant 2$ and that (P_{n-1}) holds. First, note that $\mu_{a^n}(x) = a\mu_{a^{n-1}}(x) +$

$\mu_a(x)a^{n-1}$. Therefore, $\mu_{a^n}(x) \in \Sigma(W)a$ if and only if $a\mu_{a^{n-1}}(x) \in \Sigma(W)a$. But, by Corollary 3.12, the map $\kappa : a\Sigma(W) \to a\Sigma(W)$, $u \mapsto au$ is an isomorphism and stabilizes $\Sigma(W)a$. Therefore, $a\mu_{a^{n-1}}(x) \in \Sigma(W)a$ if and only if $\mu_{a^{n-1}}(x) \in \Sigma(W)a$. In other words,

$$\operatorname{Im}\mu_{a^n} \cap \Sigma(W)a = \kappa^{-1}(\operatorname{Im}\mu_{a^{n-1}} \cap \Sigma(W)a).$$

This shows (P_n). □

3.D. Counter-examples. In this subsection, we provide examples to show that the different results of this section might fail if the positivity property is not satisfied.

• FIRST STATEMENT OF PROPOSITION 3.6 - Assume here that $W = \mathfrak{S}_3$ is of type A_2 and write $S = \{s_1, s_2\}$. Let $a = x_{s_1} - x_{s_2}$. Then $\operatorname{Rad}\Sigma(W) = \mathbb{Q}a$. Therefore, $a\Sigma(W) = \mathbb{Q}a \neq \Sigma_{\mathcal{F}_{\mathrm{eq}}(a)}(W)$.

• SECOND STATEMENT OF PROPOSITION 3.6 - Assume here that $W = \mathfrak{S}_4$ is of type A_3. Write $S = \{s_1, s_2, s_3\}$, with $s_1s_3 = s_3s_1$. Let $a = x_{s_1} - x_{s_2,s_3}$. Then $x_{s_2} - x_{s_3}$ belongs to $\Sigma(W)a$ but does not belong to $a\Sigma(W)$.

• COROLLARY 3.7 - Assume here that $W = \mathfrak{S}_3$ is of type A_2 and write $S = \{s_1, s_2\}$. Let $a = x_S - x_{s_2}$. Then $\xi_S(a) > 0$ but a is not invertible.

• COROLLARY 3.8 - Let $a \in \Sigma^+(W)$ be non-zero. Then $a\Sigma(W) + (-a)\Sigma(W) = a\Sigma(W) \neq (a + (-a))\Sigma(W) = 0$.

• PROPOSITION 3.10 AND COROLLARY 3.12 - Let $a \in \operatorname{Rad}\Sigma(W)$ be non-zero. Then $f_a(T) = T^n$ for some $n \geqslant 2$, so $f_a(T)$ is not square-free. Moreover, $\Sigma(W)a \neq \Sigma(W)a^n = 0$ and $a\Sigma(W) \neq a^n\Sigma(W) = 0$.

• COROLLARY 3.21 - Let $a \in \operatorname{Rad}\Sigma(W)$ be non-central. Then there exists $n \geqslant 2$ such that $a^n = 0$ is central.

4. SOME MORPHISMS BETWEEN SOLOMON DESCENT ALGEBRAS

4.A. Restriction morphisms. Whenever $K \subset S$, F. Bergeron, N. Bergeron, R.B. Howlett and D.E. Taylor have constructed a so-called *restriction morphism* between $\Sigma(W)$ and $\Sigma(W_K)$. They do not say that they are morphisms of algebras but this can be deduced from some of their results [BBHT, 13 and Proposition 2.6]. However, we present here a simpler proof (see Proposition 4.1).

In this subsection, we recall the definition and the basic properties of these restriction morphisms, and we prove some results on their image. We first need some notation:

> NOTATION - If $K \subset S$, we denote by X_I^K, x_I^K, θ_K, $\equiv_K$, Λ_K, $\boldsymbol{\lambda}_K$ and τ_λ^K for the objects defined in W_K instead of W and which correspond respectively to X_I, x_I, θ, $\equiv$, Λ, $\boldsymbol{\lambda}$ and τ_λ.

If $K \subset S$, let $\operatorname{Res}_K : \Sigma(W) \to \Sigma(W_K)$ denote the $\mathbb{Q}$-linear map such that

$$\operatorname{Res}_K(x_I) = \sum_{d \in X_{KI}} x^K_{K \cap {}^d I}$$

for every $I \in \mathcal{P}(S)$. If $K \subset L \subset S$, we denote by $\operatorname{Res}_K^L : \Sigma(W_L) \to \Sigma(W_K)$ the map defined like Res_K but inside W_L. Finally, if $K' \in \mathcal{P}(S)$ and if $d \in X_{KK'}$ are such that ${}^dK' = K$, then the map $d_* : \Sigma(W_{K'}) \to \Sigma(W_K)$, $x \mapsto dxd^{-1}$ is well-defined and is an isomorphism of algebras. It sends $x_I^{K'}$ to $x_{{}^dI}^K$ $(I \in \mathcal{P}(K'))$. Let us gather in the next proposition the formal properties of the map Res_K:

Proposition 4.1. *Let $K \in \mathcal{P}(S)$. Then:*

(a) *If $x \in \Sigma(W)$, then $x_K \operatorname{Res}_K(x) = xx_K$.*
(b) *Res_K is an homomorphism of algebras.*
(c) *If $K \subset L \subset S$, then $\operatorname{Res}_K = \operatorname{Res}_K^L \circ \operatorname{Res}_L$.*
(d) *The diagram*

$$\begin{tikzcd} \Sigma(W) \arrow[r, "\theta"] \arrow[d, "\operatorname{Res}_K"'] & \mathbb{Q}\operatorname{Irr} W \arrow[d, "\operatorname{Res}_{W_K}^W"] \\ \Sigma(W_K) \arrow[r, "\theta_K"] & \mathbb{Q}\operatorname{Irr} W_K \end{tikzcd}$$

is commutative.
(e) *If $K' \in \mathcal{P}(S)$ and if $d \in X_{KK'}$ are such that ${}^dK' = K$, then*

$$\operatorname{Res}_K = d_* \circ \operatorname{Res}_{K'}.$$

Proof. If $I \subset K$, then $x_K x_I^K = x_I$. So the map $\mu_K : \Sigma(W_K) \to \Sigma(W)$, $x \mapsto x_K x$ is well-defined and injective. (a) follows from this observation and from Solomon's Theorem (a). (b) and (c) follow from (a) and from the injectivity of μ_K (note that $\operatorname{Res}_K(1) = 1$). (d) is a direct consequence of the Mackey formula. Finally, we have $x_{K'} = x_K d$. So (e) follows again from (a) and from the injectivity of μ_K. $\square$

The natural map $\mathcal{P}(K) \to \mathcal{P}(S)$ induces a map $\pi_K : \Lambda_K \to \Lambda$. The next corollary is a generalisation of [A, Theorem 3.6].

Corollary 4.2. *If $\lambda \in \Lambda_K$, then $\tau_{\pi_K(\lambda)} = \tau_\lambda^K \circ \operatorname{Res}_K$.*

Proof. This follows from Proposition 4.1 (d). $\square$

The Corollary 4.2 can be written as follows: if $I \in \mathcal{P}(K)$, then

(4.3) $$\tau_{\boldsymbol{\lambda}(I)} = \tau_{\boldsymbol{\lambda}_K(I)} \circ \operatorname{Res}_K .$$

The next result generalizes [A, Theorem 2.3].

Proposition 4.4. *$\Sigma(W) = \operatorname{Ker}\operatorname{Res}_K \oplus \Sigma(W)x_K$ and $\operatorname{Ker}\operatorname{Res}_K$ is the set of $x \in \Sigma(W)$ such that $xx_K = 0$.*

Proof. By Proposition 4.1 (a), we have $\dim_{\mathbb{Q}}(\operatorname{Im}\operatorname{Res}_K) = \dim_{\mathbb{Q}} \Sigma(W)x_K$. Therefore,

$$\dim_{\mathbb{Q}}(\operatorname{Ker}\operatorname{Res}_K) + \dim_{\mathbb{Q}} \Sigma(W)x_K = \dim_{\mathbb{Q}} \Sigma(W).$$

Now, let $x \in \Sigma(W)$ be such that $xx_K \in \operatorname{Ker}\operatorname{Res}_K$. According to the previous equality, it is sufficient to show that $xx_K = 0$. But, by Proposition 4.1 (a), we have that $xx_K^2 = 0$. By Corollary 3.12, this implies that $xx_K = 0$. $\square$

Corollary 4.5. *The following are equivalent:*

(1) *Res_K is surjective.*
(2) *$\dim \Sigma(W)x_K = 2^{|K|}$.*
(3) *$\Sigma(W)x_K = \Sigma_{\mathcal{P}(K)}(W)$.*

Proof. By Proposition 4.1 (a), we have that $\dim_{\mathbb{Q}} \Sigma(W)x_K = \dim_{\mathbb{Q}}(\operatorname{Im}\operatorname{Res}_K)$. By Solomon's Theorem (a), we have that $\Sigma(W)x_K \subset \Sigma_{\mathcal{P}(K)}(W)$. Moreover, note that $\dim_{\mathbb{Q}} \Sigma_{\mathcal{P}(K)}(W) = 2^{|K|}$. The corollary follows from Proposition 4.4 and these three observations. $\square$

We now investigate further the image of Res_K. First, let

$$W(K) = \{w \in X_{KK} \mid {}^wK = K\}.$$

Then $W(K)$ is a subgroup of W and

$$N_W(W_K) = W(K) \ltimes W_K.$$

Moreover, $W(K)$ acts on $\Sigma(W_K)$ by conjugation.

Proposition 4.6. $\operatorname{Im}\operatorname{Res}_K \subset \Sigma(W_K)^{W(K)}$.

Proof. This follows immediately from Proposition 4.1 (e). □

Corollary 4.7. *If* Res_K *is surjective, then the map* $\pi_K : \Lambda_K \to \Lambda$ *is injective and* $W(K)$ *acts trivially on* W_K.

Proof. This follows immediately from Corollary 4.2 and Proposition 4.6. □

EXAMPLE 4.8 - Assume here that $W = \mathfrak{S}_n$ is the symmetric group of degree n. View $\mathfrak{S}_{n-1}$ as a parabolic subgroup of W. Then, by [BGR], the restriction morphism $\Sigma(\mathfrak{S}_n) \to \Sigma(\mathfrak{S}_{n-1})$ is surjective. Therefore, by Proposition 4.1 (c) and (e), if K is a subset of S such that W_K is irreducible, then Res_K is surjective.

Moreover, the map π_K is injective if and only if W_K is irreducible. So we have shown that, if W is irreducible of type A, then Res_K is surjective if and only if π_K is injective. □

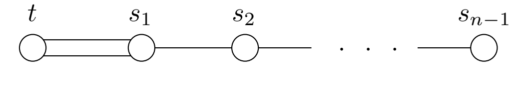
EXAMPLES 4.9 - Let W be irreducible of exceptional type. Let $n = |S|$. We write $S = \{s_1, s_2, \ldots, s_n\}$ following the convention of Bourbaki [Bbk, Planches I-IX]. For simplification, we denote by $i_1 i_2 \ldots i_k$ the subset $\{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}$ of S (for instance, 134 stands for $\{s_1, s_3, s_4\}$). Then, computations using `CHEVIE` show that:

(a) If W is of type E_6, E_7, E_8, G_2 or H_3, then Res_K is surjective if and only if $|K| \in \{0, 1, |S|\}$.
(b) If W is of type F_4, then Res_K is surjective if and only if K belongs to $\{1234, 123, 234, 13, 14, 23, 24, 1, 2, 3, 4, \varnothing\}$.
(c) If W is of type H_4, then Res_K is surjective if and only if K belongs to $\{1234, 123, 1, 2, 3, 4, \varnothing\}$. □

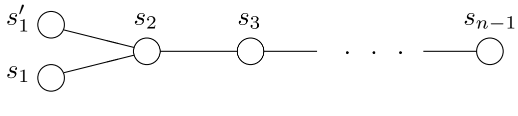
REMARK - The examples 4.9 show that the converse of Corollary 4.7 is not true in general. □

We will see in the next subsection some other examples of restriction morphisms (groups of type B or D) and some results concerning their images.

4.B. Type B, type D: another restriction morphism. We shall investigate here some properties of $\Sigma(W)$ whenever W is of type B or D. We fix in this subsection a natural number $n \geqslant 1$. Let (W_n, S_n) be a Coxeter group of type B_n. We write $S_n = \{t, s_1, s_2, \ldots, s_{n-1}\}$ in such a way that the Dynkin diagram of W_n is



Let $s_1' = ts_1t$, $S_n' = \{s_1', s_1, s_2, \ldots, s_{n-1}\}$ and $W_n' =< S_n' >$. Then (W_n', S_n') is a Weyl group of type D_n: its Dynkin diagram is



Recall that $W_n = < t > \ltimes W'_n$. So $X_n = \{1, t\}$ is the set of minimal length coset representatives of W_n/W'_n. We set $x_n = 1 + t \in \mathbb{Q}W_n$. Note that conjugacy by t induces the unique non-trivial automorphism of W'_n which stabilizes S'_n: this automorphism will be denoted by σ_n. If $I \subset S'_n$ or if $I \subset S_n$, we denote by W_I the subgroup of W_n generated by I. It is a standard parabolic subgroup of W'_n or of W_n and it might be a parabolic subgroup of both. If $I \subset S'_n$, we still denote by $X_I^{S_n}$ the set of $w \in W_n$ such that w has minimal length in wW_I and we set $x_I^{S_n} = \sum_{w \in X_I^{S_n}} w \in \mathbb{Q}W_n$. Therefore, if $I \subset S'_n$,

(4.10) $$x_I^{S_n} = (1+t)x_I^{S'_n}.$$

If $I \subset S_n$, then it is easy to check that

(4.11) $$W_I \cap W'_n = W_{W_I \cap S'_n} \qquad \text{and} \qquad {}^tW_I \cap W'_n = W_{{}^tW_I \cap S'_n}.$$

Moreover, if $t \notin I$, then

(4.12) $$W_I \cap W'_n = W_I \qquad \text{and} \qquad {}^tW_I \cap W'_n = W_{{}^tI}.$$

We set
$$X_{I,n} = X_n \cap X_I^{-1}$$
and
$$\mathrm{Res}_n\, x_I^{S_n} = \sum_{d \in X_{I,n}} x^{S'_n}_{{}^{d^{-1}}W_I \cap S'_n} \in \Sigma(W'_n).$$
In other words, by 4.11 and 4.12,

(4.13) $$\mathrm{Res}_n\, x_I^{S_n} = \begin{cases} x^{S'_n}_{W_I \cap S'_n} & \text{if } t \in I, \\ x_I^{S'_n} + x_{{}^tI}^{S'_n} & \text{if } t \notin I. \end{cases}$$

This can be extended by linearity to a map $\mathrm{Res}_n : \Sigma(W_n) \to \Sigma(W'_n)$. This map shares with the restriction morphisms many properties:

Proposition 4.14. *With the above notation, we have:*

(a) *If $x \in \Sigma(W_n)$, then $x_n \mathrm{Res}_n(x) = xx_n$.*
(b) Res_n *is an homomorphism of algebras.*
(c) $\mathrm{Res}^{S'_n}_{S'_{n-1}} \circ \mathrm{Res}_n = \mathrm{Res}_{n-1} \circ \mathrm{Res}^{S_n}_{S_{n-1}}$.
(d) *The diagram*

$$\begin{tikzcd} \Sigma(W_n) \arrow[r, "\theta_n"] \arrow[d, "\mathrm{Res}_n"'] & \mathbb{Q}\,\mathrm{Irr}\, W_n \arrow[d, "\mathrm{Res}^{W_n}_{W'_n}"] \\ \Sigma(W'_n) \arrow[r, "\theta'_n"] & \mathbb{Q}\,\mathrm{Irr}\, W'_n \end{tikzcd}$$

is commutative.

(e) $\mathrm{Im}\,\mathrm{Res}_n = \Sigma(W'_n)^{\sigma_n}$.

Proof. (a) Let $I \subset S_n$. We want to prove that $x_I^{S_n}(1+t) = (1+t)\,\mathrm{Res}_n(x_I^{S'_n})$. First, assume that $t \notin I$. Then $W_I \subset W'_n$. Therefore, $x_I^{S_n} = (1+t)x_I^{S'_n}$. Consequently,

$$\begin{aligned} x_I^{S_n}(1+t) &= (1+t)x_I^{S'_n}(1+t) \\ &= x_I^{S'_n} + tx_I^{S'_n} + x_I^{S'_n}t + tx_I^{S'_n}t \\ &= (1+t)(x_I^{S'_n} + x_{{}^tI}^{S'_n}) \\ &= (1+t)\,\mathrm{Res}_n(x_I^{S_n}), \end{aligned}$$

as desired. Now, assume that $t \in I$. Then $X_n = \{1, t\}$ is a set of minimal length coset representatives of $W_I/(W_I \cap W'_n)$. Therefore $x_I^{S_n} x_n = x_{W_I \cap S'_n}^{S_n} = x_n x_{W_I \cap S'_n}^{S'_n}$, as expected (note that the last equality follows from 4.10.

(b) First, note that $\mathrm{Res}_n(1) = \mathrm{Res}_n(x_{S_n}^{S_n}) = x_{S'_n}^{S'_n} = 1$ by definition. The fact that $\mathrm{Res}_n(xy) = \mathrm{Res}_n(x)\,\mathrm{Res}_n(y)$ for all x, $y \in \Sigma(W_n)$ follows immediately from (a) and from the fact that the map $\mu_n : \mathbb{Q}W'_n \to \mathbb{Q}W_n$, $x \mapsto x_n x$ is injective.

(c) follows also from (a) and from the fact $x_n x_{S'_{n-1}}^{S'_n} = x_{S_{n-1}}^{S_n} x_{n-1}$.

(d) follows from the Mackey formula for tensor product of induced characters.

(e) This follows easily from 4.13. □

We conclude this subsection by two examples where the image of the restriction map Res_K is computed explicitly. The first one concerns type B (see Proposition 4.15) while the second one concerns the type D (see Corollary 4.16).

Proposition 4.15. *The map* $\mathrm{Res}_{S_{n-1}}^{S_n} : \Sigma(W_n) \to \Sigma(W_{n-1})$ *is surjective.*

Proof. We have

$$X_{S_{n-1}}^{S_n} = \{s_i s_{i+1} \dots s_{n-1} \mid 1 \leqslant i \leqslant n\} \coprod \{s_i s_{i-1} \dots s_1 t s_1 s_2 \dots s_{n-1} \mid 0 \leqslant i \leqslant n-1\}.$$

Therefore, if $d \in W_n$ and if $i \in \{1, 2, \dots, n-1\}$ are such that $d^{-1} \in X_{S_{n-1}}^{S_n}$, $ds_i > d$, and $ds_i d^{-1} \in S_{n-1}$, then

$$(*) \qquad ds_i d^{-1} \in \{s_i, s_{i-1}\}.$$

We define a total order $\preccurlyeq$ on $\mathcal{P}(S_{n-1})$. Let I and J be two subsets of S_{n-1}. Then we write $I \preccurlyeq J$ if and only if one of the following two conditions are satisfied:

(1) $|I| < |J|$
(2) $|I| = |J|$ and I is smaller than J for the lexicographic order on $\mathcal{P}(S_{n-1})$ induced by the order $t < s_1 < \cdots < s_{n-1}$ on S_{n-1}.

It follows immediately from $(*)$ that

$$\mathrm{Res}_{S_{n-1}}^{S_n} x_J^{S_n} \in \alpha_J x_J^{S_{n-1}} + \sum_{I \prec J} \mathbb{Q} x_I^{S_{n-1}}$$

with $\alpha_J > 0$ (for every $J \in \mathcal{P}(S_{n-1})$). The proof of the proposition is complete. □

Corollary 4.16. *The image of the map* $\mathrm{Res}_{S'_{n-1}}^{S'_n} : \Sigma(W'_n) \to \Sigma(W'_{n-1})$ *is equal to* $\Sigma(W'_{n-1})^{\sigma_{n-1}}$.

Proof. This follows from Proposition 4.14 (c) and (e) and from Proposition 4.15. □

REMARK 4.17 - If n is odd, then $\sigma_n = \sigma_0$, the automorphism of $\Sigma(W'_n)$ induced by conjugation by the longest element of W'_n. □

4.C. Self-opposed subsets. A subset K of S is called *self-opposed* if, for every $w \in W$ such that ${}^wK \subset S$, we have ${}^wK = K$.

In this subsection, we fix a self-opposed subset K of S. If $s \in S \setminus K$, we set $w_{K,s} = w_{K \cup \{s\}} w_K$ (here, if I is a subset of S, w_I denotes the longest element of W_I). Then, since K is self-opposed, we have $w_{K,s} \in W(K)$. Now, if I is a subset

of S containing K, we set $I(K) = \{w_{K,s} \mid s \in I \setminus K\}$. Then (see for instance [GP, Remark 2.3.5])

(4.18) $(W(K), S(K))$ *is a finite Coxeter group.*

> NOTATION - We denote by $X_I^{(K)}$, $x_I^{(K)}$, $\Lambda_{(K)}$ and $\boldsymbol{\lambda}_{(K)}$ the objects defined like X_I, x_I, Λ or $\boldsymbol{\lambda}$ but inside $W(K)$.

Let $\psi_K : \Sigma(W) \to \Sigma(W(K))$ be the linear map such that

$$\psi_K(x_I) = \begin{cases} x_{I(K)}^{(K)} & \text{if } K \subset I, \\ 0 & \text{otherwise,} \end{cases}$$

for every subset I of S.

Proposition 4.19. *Assume K is self-opposed in S. Let $I, J, L \subseteq S$ be such that $K \subseteq I, J, L$. Then $X_{I(K)J(K)L(K)}^{(K)} = X_{IJL}$.*

Proof. First note that, if $l_{(K)}$ is the length function of $W(K)$ with respect to $S(K)$ then, for any $s \in S \setminus K$, we have that $l(ws) > l(w)$ if and only if $l_{(K)}(ww_{K,s}) > l_{(K)}(w)$ (see [L, Theorem 5.9]). It follows that $X_{J(K)}^{(K)} = X_J \cap W(K)$ for every subset J of S containing K. Moreover, $W_{J(K)} = W_J \cap W(K)$.

Let $d \in X_{IJL}$ for some $L \subseteq S$ containing K. Then $K \subseteq L \subseteq I^d$ implies $d \in W(K)$. Also $W_I^d \cap W_J = W_L$ implies $W_{I(K)}^d \cap W_{J(K)} = (W_I \cap W(K))^d \cap W_J \cap W(K) = W_L \cap W(K) = W_{L(K)}$, whence $X_{IJL} \subseteq X_{I(K)J(K)L(K)}^{(K)}$ for all $K \subseteq L \subseteq S$.

Equality follows from that fact that $X_{I(K)J(K)}^{(K)} = X_{IJ} \cap W(K)$ is both the disjoint union of the sets $X_{I(K)J(K)L(K)}^{(K)}$ with $K \subseteq L \subseteq S$ and the disjoint union of the sets X_{IJL} with $K \subseteq L \subseteq S$. $\square$

Theorem 4.20. *If K is a self-opposed subset of S, then ψ_K is a surjective homomorphism of algebras.*

Proof. The surjectivity of ψ_K is clear from the definition. Also, $\psi_K(1) = \psi_K(x_S) = x_{S(K)} = 1$. Let us now prove that ψ_K is respects the multiplication. Let I, J be two subsets of S. We want to prove that

$$(*) \qquad \psi_K(x_I x_J) = \psi_K(x_I)\psi_K(x_J).$$

Assume first that I (or J) does not contain K. Then $\psi_K(x_I)\psi_K(x_J) = 0$. Let $d \in X_{IJ}$. If K is contained in ${}^{d^{-1}}I \cap J$, then K is contained in J or ${}^dK$ is contained in $I \subset S$, so K is contained in I or in J, which is impossible. So $\psi_K(x_I x_J) = 0$.

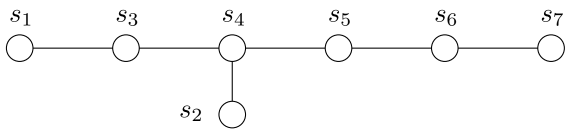
Assume now that both I and J contain K. Then

$$\psi_K(x_I x_J) = \sum_{K \subset L \subset S} |X_{IJL}|\, x_{L(K)}^{(K)}.$$

In this case, $(*)$ follows from Proposition 4.19. $\square$

EXAMPLE 4.21 - Assume here that W is of type B_n and keep the notation of the proof of Proposition 4.15. Then $\{t\}$ is a self-opposed subset of S and $W(\{t\})$ is of type B_{n-1}. So Theorem 4.20 gives another surjective morphism between the Solomon algebra of type B_n and the Solomon algebra of type B_{n-1}. This homomorphism does not coincide with the one constructed in Proposition 4.15. $\square$

EXAMPLE 4.22 - Assume here that (W, S) is of type E_7 and assume that $S = \{s_i \mid 1 \leqslant i \leqslant 7\}$ is numbered as in [Bbk, Planche VI]. In other words, the Dynkin diagram of W is:



Let $K = \{s_2, s_5, s_7\}$. Then K is self-opposed and $W(K)$ is of type F_4. So Theorem 4.20 realizes the Solomon algebra of type F_4 as a quotient of the Solomon algebra of type E_7. $\square$

If I is a subset of $S(K)$, we denote by $\varpi_K(I)$ the unique subset A of S containing K such that $A(K) = I$. Then the map $\varpi_K : \mathcal{P}(S(K)) \to \mathcal{P}(S)$ induces a map $\tilde{\varpi}_K : \Lambda_{(K)} \to \Lambda$ (indeed, by the definition of $W(K)$, if I and J are two subsets of $S(K)$ and if $w \in W(K)$ is such that ${}^wI = J$, then ${}^w\varpi_K(I) = \varpi_K(J)$). Then, if $I \subset S(K)$, we have, by 1.4,

(4.23)
$$\tau_{\boldsymbol{\lambda}(\varpi_K(I))} = \tau^{(K)}_{\boldsymbol{\lambda}_{(K)}(I)} \circ \psi_K.$$

We close this subsection by showing that the morphisms Res_L and ψ_K are compatible. More precisely, let L be a subset of S containing K. Then K is self-opposed for W_L and $W_L(K)$ is the parabolic subgroup of $W(K)$ generated by $L(K)$. Let $\psi_K^L : \Sigma(W_L) \to \Sigma(W_L(K))$ be the morphism defined like ψ_K but inside W_L. Then the diagram

(4.24)
$$\begin{tikzcd} \Sigma(W) \arrow[r, "\psi_K"] \arrow[d, "\mathrm{Res}_L"'] & \Sigma(W(K)) \arrow[d, "\mathrm{Res}_{L(K)}"] \\ \Sigma(W_L) \arrow[r, "\psi_K^L"] & \Sigma(W_L(K)) \end{tikzcd}$$

is commutative. Indeed, if I is a subset of L containing K, we have $\psi_K(x_I) = x_{I(K)} = x_{L(K)}\psi_K^L(x_I^L)$. In other words, $\psi_K(x_L x) = x_{L(K)}\psi_K^L(x)$ for every $x \in \Sigma(W_L)$. So the commutativity of 4.24 follows from Proposition 4.1 (a) and from routine computations.

5. LOEWY LENGTH OF $\Sigma(W)$

The *Loewy length* of a finite dimensional algebra A is the smallest natural number $k \geqslant 1$ such that $(\mathrm{Rad}\, A)^k = 0$. We denote by $\mathrm{LL}(W)$ the Loewy length of $\Sigma(W)$. If σ is an automorphism of W such that $\sigma(S) = S$, we denote by $\mathrm{LL}(W, \sigma)$ the Loewy length of $\Sigma(W)^\sigma$. By Corollary 2.2, we have

(5.1)
$$\mathrm{LL}(W, \sigma) \leqslant \mathrm{LL}(W).$$

By Solomon's Theorem (e), $\mathrm{LL}(W)$ is the smallest natural number $k \geqslant 1$ such that $(\mathrm{Ker}\, \theta)^k = 0$.

5.A. Upper bound. Let us start with an easy observation (recall that σ_0 denotes the automorphism of W induced by conjugacy by w_0):

Lemma 5.2. *Let $k \geqslant 0$. Then:*

(a) $(\mathrm{Ker}\, \theta).\Sigma_k(W) \subset \Sigma_{k-1}(W)$.
(b) $(\mathrm{Ker}\, \theta)^{\sigma_0}.\Sigma_k(W) \subset \Sigma_{k-2}(W)$.

Proof. Let $J \in \mathcal{P}(S)$ be such that $|J| \leqslant k$ and let $x \in \operatorname{Ker}\theta$. Then $\tau_{\boldsymbol{\lambda}(J)}(x) = 0$. By 1.4, we then have $xx_J \in \Sigma_{k-1}(W)$, whence (a). If moreover $x \in (\operatorname{Ker}\theta)^{\sigma_0}$, then $xx'_J \in \Sigma_{k-2}(W)$ by Lemma 2.13. This shows (b). □

REMARK - It is not true in general that $\Sigma_k(W).(\operatorname{Ker}\theta) \subset \Sigma_{k-1}(W)$. □

Corollary 5.3. *We have:*

(a) $\operatorname{LL}(W) \leqslant |S|$.

(b) $\operatorname{LL}(W, \sigma_0) \leqslant \dfrac{|S|+1}{2}$.

Proof. (a) We have $\operatorname{Ker}\theta \subset \Sigma_{|S|-1}(W)$ and $\operatorname{Ker}\theta \cap \Sigma_0(W) = 0$ (see Solomon's Theorem (d)). So, by Lemma 5.2 (a), we have $(\operatorname{Ker}\theta)^{|S|} = 0$.

(b) By Lemma 5.2 (b), we have $(\operatorname{Ker}\theta)^{\sigma_0} \subset \Sigma_{|S|-2}(W)$ and $\big((\operatorname{Ker}\theta)^{\sigma_0}\big)^r \subset \Sigma_{|S|-2r}(W)$ for every $r \geqslant 0$. This shows (b). □

EXAMPLE 5.4 - It is a classical result [A, Corollary 3.5] that, if W is of type A_n, then $\operatorname{LL}(W) = n$. In this case, we also have $\operatorname{LL}(W, \sigma_0) = \left\lceil \dfrac{n}{2} \right\rceil$. Indeed, let $l = \operatorname{LL}(W, \sigma_0)$. By Corollary 5.3 (b), we have $l \leqslant \left\lceil \dfrac{n}{2} \right\rceil$. On the other hand, let $a = x_{\{s_1,\dots,s_{n-1}\}} - x_{\{s_2,\dots,s_n\}}$, where $S = \{s_1, s_2, \dots, s_n\}$ is numbered such that $(s_i s_{i+1})^3 = 1$ for every $i \in \{1, 2, \dots, n-1\}$. Then, by [A, Proof of Corollary 3.5], we have $a \in \operatorname{Rad}\Sigma(W)$ and $a^{n-1} \neq 0$. In particular, $(a^2)^{\left\lceil \frac{n-1}{2} \right\rceil} \neq 0$. But, $\sigma_0(a) = -a$, so $\sigma_0(a^2) = a^2$. Therefore, by Corollary 2.2, we have $a^2 \in \operatorname{Rad}\big(\Sigma(W)^{\sigma_0}\big)$. So $l \geqslant \left\lceil \dfrac{n}{2} \right\rceil$, as desired. □

5.B. Type B. We keep the notation of subsection 4.B. The aim of this subsection is to prove the next proposition:

Proposition 5.5. *If* $n \geqslant 1$, *then* $\operatorname{LL}(W_n) = \left\lceil \dfrac{n}{2} \right\rceil$.

Proof. By Corollary 5.3 (b), we have $\operatorname{LL}(W_n) \leqslant \left\lceil \dfrac{n}{2} \right\rceil$. Now, let $r = \left\lceil \dfrac{n-1}{2} \right\rceil$. It is sufficient to find $a_1, \dots, a_r \in \operatorname{Rad}\Sigma(W_n)$ such that $a_r \dots a_1 \neq 0$.

If $1 \leqslant i \leqslant j \leqslant n-1$, we set $[i,j] = \{s_i, s_{i+1}, \dots, s_j\}$. If $1 \leqslant i \leqslant r$, we set

$$a_i = x_{[2i-1,n-2]} - x_{[2i,n-1]}.$$

Then $a_i \in \operatorname{Rad}\Sigma(W_n)$. We shall show that $a_r \dots a_1 \neq 0$. If $1 \leqslant i \leqslant r$, we set

$$\tau_i = \sum_{j=0}^{2i-1} (-1)^j \binom{2i-1}{j} x_{[j+1,n-2i+j]}.$$

We will show by induction on i that

$$(P_i) \qquad a_i \dots a_1 \in \mathbb{Q}^\times \tau_i + \Sigma_{n-2i-1}(W_n).$$

Note that, if (P_r) is proved, then the proposition is complete. Now, (P_1) holds since $a_1 = \tau_1$. So, let $i \in \{2, 3, \dots, r\}$ and assume that (P_{i-1}) holds. By Lemma 5.2 (b), there exists three elements α, β and γ of $\mathbb{Q}$ such that

$$\begin{aligned} &a_i x_{[j+1,n-2(i-1)+j]} \\ &\qquad \in \alpha x_{[j+1,n-2i+j]} + \beta x_{[j+2,n-2i+j+1]} + \gamma x_{[j+3,n-2i+j+2]} \\ &\qquad\qquad + \Sigma_{n-2i-1}(W) \end{aligned}$$

for every $j \in \{1, 2, \dots 2i-1\}$. The fact that α, β and γ do not depend on j follows from the fact that there exists $w \in X_{[j+1,n-2(i-1)+j],[j'+1,n-2(i-1)+j']}$ such

that ${}^{w}[j'+1, n-2(i-1)+j'] = [j+1, n-2(i-1)+j]$. In particular, we have $x_{[j+1,n-2(i-1)+j]}w = x_{[j'+1,n-2(i-1)+j']}$. Since $a_i \in \operatorname{Ker}\theta$, we have $\alpha+\beta+\gamma = 0$. Also,

$$x_{[j+1,n-2(i-1)+j]}w_0 w_{[j+1,n-2(i-1)+j]} = x_{[j+1,n-2(i-1)+j]}.$$

Therefore, $\alpha = \gamma$. In other words,

$$\begin{aligned} a_i x_{[j+1,n-2(i-1)+j]} & \\ \in \alpha(x_{[j+1,n-2i+j]} - 2x_{[j+2,n-2i+j+1]} + x_{[j+3,n-2i+j+2]}) & \\ + \Sigma_{n-2i-1}(W). & \end{aligned}$$

Hence, by the induction hypothesis, by Lemma 5.2 (b) and by usual properties of binomial coefficients, we have

$$a_i \dots a_1 \in \mathbb{Q}^\times \alpha\tau_i + \Sigma_{n-2i-1}(W_n).$$

So it remains to show that

$$\alpha \neq 0.$$

For this, consider the case where $j = 2i-3$ and write

$$x_{[2i-1,n-2]}x_{[2i-2,n-1]} = ax_{[2i-2,n-3]} + bx_{[2i-1,n-2]} + cx_{[2i,n-1]}$$

and
$$x_{[2i,n-1]}x_{[2i-2,n-1]} = dx_{[2i-2,n-3]} + ex_{[2i-1,n-2]} + fx_{[2i,n-1]}$$

with a, b, c, d, e and f in $\mathbb{Q}$. We then have $b-e = -2\alpha$. Since $b \neq 0$, it is sufficient to show that $e = 0$. In other words, we need to prove the following lemma:

Lemma 5.6. *If $d \in X_{[2i,n-1],[2i-2,n-1]}$, then ${}^{d^{-1}}[2i, n-1] \neq [2i-1, n-2]$.*

Proof of Lemma 5.6. We identify W_n with the group of permutations σ of $E = \{\pm1, \pm2, \dots, \pm n\}$ such that $\sigma(-k) = -\sigma(k)$ for every $k \in E$ (t corresponds to the transposition $(-1, 1)$ while s_k corresponds to $(k, k+1)(-k, -k-1)$). If $d \in X_{[2i-2,n-1]}$, then d is increasing on $\{2i-2, 2i-1, \dots, n-2, n-1\}$. If moreover ${}^{d^{-1}}[2i, n-1] = [2i-1, n-2]$ (in other words, if ${}^{d}[2i-1, n-2] = [2i, n-1]$), then $d(\{2i-1, 2i, \dots, n-1\}) \subset \{\pm 2i, \pm(2i+1), \cdots \pm n\}$ and d has constant sign on $\{2i-1, 2i, \dots, n-1\}$. Two cases may occur:

• If $d(2i-1) > 0$, then, since d has constant sign and is increasing on $\{2i-1, 2i, \dots, n-1\}$, we have $d(n-1) = n$. But this is impossible since $d(n) > d(n-1)$.

• If $d(2i-1) < 0$, then, by the same argument, we have $d(2i-1) = -n$. But this is again impossible since $d(2i-2) < d(2i-1)$. □

The proof of (P_i) and of the proposition is now complete. □

Remark 5.7 - Assume here that W is of type B_{2r+1}, $r \geqslant 1$ and let τ_r denote the element of $\Sigma(W)$ defined in the proof of Proposition 5.10. Computations using `CHEVIE` show that the following question has a positive answer for $m \in \{1, 2, 3\}$:

Question: *Is it true that* $(\operatorname{Ker}\theta)^r = \mathbb{Q}\tau_r$*?* □

5.C. Type D. The following result is an easy consequence of Proposition 5.5 (and its proof) and of the existence of the homomorphism of algebras Res_n.

Corollary 5.8. *If $n \geqslant 1$, then* $\operatorname{LL}(W_n', \sigma_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let l be the Loewy length of $\Sigma(W_n')^{\sigma_n}$. Keep the notation of the proof of Proposition 5.5. Let $a_i' = \operatorname{Res}_n a_i$. Then, by Proposition 4.14 (b) and (e), we have $a_i' \in \operatorname{Rad}\Sigma(W_n')^{\sigma_n}$ and $a_r \dots a_1 = \alpha \operatorname{Res}_n \tau_r$, where $\alpha \neq 0$. But, it is clear from the definition of Res_n that $\operatorname{Res}_n \tau_r \neq 0$. So $a_r' \dots a_1' \neq 0$. So $l \geqslant r+1$.

The fact that $l \leqslant r+1$ follows from Propositions 5.5 and 4.14 (e). $\square$

Corollary 5.9. *Let $n \geqslant 3$. Then:*

(a) *If n is even, then* $\mathrm{LL}(W_n') = \left\lceil \dfrac{n}{2} \right\rceil$.

(b) *If n is odd, then* $\mathrm{LL}(W_n') \geqslant \dfrac{n+3}{2}$.

Proof. By Proposition 4.14 (e),

$$(*) \qquad \operatorname{Res}_n(\operatorname{Rad}\Sigma(W_n)) = \operatorname{Rad}\Sigma(W_n')^{\sigma_n} = (\operatorname{Rad}\Sigma(W_n'))^{\sigma_n}.$$

So (a) follows from $(*)$, from Corollary 5.3 and from Corollary 5.8.

Let us now prove (b). Write $n = 2r+1$ and keep the notation of the proof of Corollary 5.8. Let $a = x_{\{s_1', s_2, \dots, s_{2r}\}} - x_{\{s_1, s_2, \dots, s_{2r}\}}$. An easy computation shows that

$$a_1' x_{\{s_1, s_2, \dots, s_{2r}\}} \in a_1' + \Sigma_{\mathcal{P}\#(\{s_1, s_2, \dots, s_{2r}\})}(W).$$

But, by the equalities (3) and (4) of the proof of Proposition 5.10 and by Lemma 5.2 (b), we have $\sigma_r \dots \sigma_2 \Sigma_{\mathcal{P}\#(\{s_1, s_2, \dots, s_{2r}\})}(W) = 0$. Therefore,

$$\sigma_r \dots \sigma_1 x_{\{s_1, s_2, \dots, s_{2r}\}} = \sigma_r \dots \sigma_1.$$

Since $x_{\{s_1', s_2, \dots, s_{2r}\}} = x_{\{s_1, s_2, \dots, s_{2r}\}} d$, where $d = w_{[1,r]} w_0$, we get that

$$\sigma_r \dots \sigma_1 a = \sigma_r \dots \sigma_2 \sigma_1 (1-d).$$

Therefore, $\sigma_r \dots \sigma_1 a \neq 0$ (indeed, the coefficient of $x_{s_1'}$ is non-zero). But, $a \in \operatorname{Ker}\theta$ because $|S|$ is odd. So $\mathrm{LL}(W_{2r+1}') \geqslant r+2$, as desired. $\square$

5.D. Lower bound. The aim of this subsection is to prove the following result:

Proposition 5.10. *If W is irreducible, then* $\mathrm{LL}(W) \geqslant \mathrm{LL}(W, \sigma_0) \geqslant \left\lceil \dfrac{|S|}{2} \right\rceil$.

Proof. By 5.1, we only need to prove the second inequality. The proof of this proposition will proceed by a case-by-case analysis. First, the exceptional groups can be treated by using `CHEVIE` (see the Table given at the end of this paper). If W is of type A, then this follows from Example 5.4. If $|S| = 2$, then there is nothing to prove. If W is of type B, this follows from Proposition 5.5. If W is of type D, this follows from Corollary 5.9. The proof is complete. $\square$

The next result follows from Corollary 5.3 and Proposition 5.10

Corollary 5.11. *If W is irreducible, then* $\mathrm{LL}(W, \sigma_0) = \left\lceil \dfrac{|S|}{2} \right\rceil$.

Corollary 5.12. *If W is irreducible and w_0 is central in W, then* $\mathrm{LL}(W) = \left\lceil \dfrac{|S|}{2} \right\rceil$.

5.E. Conclusion. The next table gives the known Loewy lengths of the algebras $\Sigma(W)^\sigma$ for W irreducible and σ is a length-preserving automorphism of W.

Type of W	$o(\sigma)$	$\lvert\Lambda/\sigma\rvert$	$\mathrm{LL}(W,\sigma)$	$d_0, d_1, d_2, \ldots$
A_n	1	$p(n)$	n	
	2	$p(n)$	$\left\lceil \frac{n}{2} \right\rceil$	
B_n	1	$\sum_{r=0}^{n} p(r)$	$\left\lceil \frac{n}{2} \right\rceil$	
D_{2n}	1	$p(n) + p(2n) + \sum_{r=0}^{2n-2} p(r)$	n	
$(n \geqslant 2)$	2	$p(2n) + \sum_{r=0}^{2n-2} p(r)$	n	
D_{2n+1}	1	$p(2n) + \sum_{r=0}^{2n-2} p(r)$	$\geqslant n+2$ (*)	
$(n \geqslant 2)$	2	$p(2n) + \sum_{r=0}^{2n-2} p(r)$	$n+1$	
D_4	1	11	2	$16, 5$
	2	9	2	$12, 3$
	3	7	2	$8, 1$
E_6	1	17	5	$64, 47, 28, 12, 3$
	2	17	3	$40, 23, 5$
E_7	1	32	4	$128, 96, 34, 2$
E_8	1	41	4	$256, 215, 106, 14$
F_4	1	12	2	$16, 4$
	2	8	2	$10, 2$
H_3	1	6	2	$8, 2$
H_4	1	10	2	$16, 6$
$I_2(2m)$	1	4	1	4
	2	3	1	3
$I_2(2m+1)$	1	3	2	$4, 1$
	2	3	1	3

The exceptional groups are obtained by using `CHEVIE`. Type A is mainly due to Atkinson [A, Corollary 3.5] (see Example 5.4). Types B and D are done in this paper (except for the type D_{2n+1}). Dihedral groups are easy. It must be noticed that the inequality $(*)$ is an equality for $n = 2$ or 3. We suspect it is always an equality.

In this table, d_i denotes the dimension of $\big(\mathrm{Rad}(\Sigma(W)^\sigma)\big)^i$. These numbers are not given for infinite series of type A, B or D. Note that $d_0 = \dim \Sigma(W)^\sigma$ and that $|\Lambda/\sigma| = d_0 - d_1$. We denote by $p(n)$ the number of partitions of n. We denote by $o(\sigma)$ the order of σ: it characterizes the conjugacy class of σ in the group of automorphism of W stabilizing S.

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