DISTRIBUTIVE COSET GRAPHS OF FINITE COXETER GROUPS

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Abstract. Let W be a finite Coxeter group, W_J a parabolic subgroup of W and X_J the set of distinguished coset representatives of W_J in W equipped with the induced weak Bruhat ordering of W. All instances when X_J is a distributive lattice are known. In this note we present a new short conceptual proof of this result.

1. Introduction.

Throughout, W denotes a finite Coxeter group, generated by a set of simple reflections $S \subseteq W$. For $J \subseteq S$, let W_J be the parabolic subgroup of W generated by J. It is well-known that W is a lattice when equipped with the weak Bruhat order [3, Thm. 8], see Lemma 2.1. Let X_J denote the set of distinguished (right) coset representatives of W_J in W endowed with the induced (right) weak Bruhat ordering from W. All instances when X_J is a distributive lattice are known:

1.1. **Theorem.** *Let W be a finite irreducible Coxeter group and let $J \subsetneq S$. Then X_J is a distributive lattice if and only if one of the following holds:*

(i) *W is a Weyl group and W_J is minuscule;*
(ii) *W is dihedral and W_J is of type A_1;*
(iii) *W is of type H_3 and W_J is of type $I_2(5)$.*

In case W is a Weyl group we say that W_J is minuscule provided W_J is the stabiliser of a minuscule weight of W.

In case W is a Weyl group Theorem 1.1 was first proved by R.A. Proctor [7, Prop. 3.1, 3.2] in a case by case analysis. More specifically, in [7, Prop. 3.2] Proctor lists each X_J from Theorem 1.1 as the poset of all order ideals of some explicit poset. Thanks to [9, Thm. 3.4.1] the latter are known to be distributive lattices. In [10, Thm. 7.1] J.R. Stembridge uses his theory of fully commutative elements to prove Theorem 1.1. Moreover, Stembridge shows in *loc. cit.* that X_J is a distributive lattice when endowed with the (right) weak Bruhat order if and only if it is a distributive lattice when endowed with the strong Bruhat order. In [7] Proctor uses the strong Bruhat order.

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In this note we give a new proof of Theorem 1.1 utilising a Mackey type induction formula involving the structure constants of the descent algebra of W. Our approach is new and avoids case by case considerations as far as possible, showing that in all cases not listed above, X_J is not distributive. We do make use of the classification of the irreducible Coxeter groups and the structure of the root systems of Weyl groups.

After a preliminary section we interpret the property that X_J is not distributive in terms of the positivity of certain structure constants of the descent algebra of W. In Section 3 we derive inductive tools for the positivity of the relevant structure constants. Our proof of Theorem 1.1 is based on this study.

Let Γ_J be the coset graph of the poset X_J, i.e., the corresponding Hasse diagram. Continuing to investigate the structure constants of the descent algebra of W we obtain the surprising consequence that for each simple reflection $s \in S$, the coset graph $\Gamma_{\{s\}}$ contains the full Cayley graph of *every* maximal parabolic subgroup of W as a subgraph; see Corollary 3.11.

Concerning the importance and ubiquity of minuscule lattices in Lie theory and combinatorial theory, see for instance [7, §11].

2. Notation and Preliminaries.

We maintain the notation from the introduction. For $W = \langle S \rangle$ a finite Coxeter group, $|S|$ is its rank. For $J \subseteq S$, denote by $X_J = \{x \in W \mid l(wx) = l(w) + l(x) \text{ for all } w \in W_J\} = \{x \in W \mid l(sx) > l(x) \text{ for all } s \in J\}$ the set of distinguished right coset representatives of W_J in W. Then, for $K \subseteq S$, the set X_K^{-1} is a set of distinguished left coset representatives of W_K in W, and $X_{JK} = X_J \cap X_K^{-1}$ is a set of double coset representatives of W_J and W_K in W. Denote $a_{JKL} = |X_{JKL}|$, where $X_{JKL} = \{d \in X_{JK} \mid J^d \cap K = L\} = \{d \in X_{JK} \mid W_J^d \cap W_K = W_L\}$ for $L \subseteq S$. Further, let $x_J = \sum_{x \in X_J} x^{-1} \in \mathbb{Q}W$, where $\mathbb{Q}W$ denotes the group algebra of W over $\mathbb{Q}$. Then, by a theorem of Solomon,

$$x_J x_K = \sum_{L \subseteq S} a_{JKL} x_L$$

(see [8, Thm. 1] or [6, (2.1.10)]). The set $\{x_J \mid J \subseteq S\}$ thus forms a basis of a subalgebra of $\mathbb{Q}W$, the *descent algebra* of W.

By $\preceq$ we denote the weak (right) Bruhat order on W, i.e., for $x, y \in W$ we have $x \preceq y$ if x is a prefix of y: $l(y) = l(x) + l(x^{-1}y)$. Thanks to [3, Thm. 8], the poset $(W, \preceq)$ is a *lattice*, i.e., any two elements $x, y \in W$ have a greatest lower and a least upper bound. Since X_J is an interval in $(W, \preceq)$, the subposet $(X_J, \preceq)$, obtained by restricting $\preceq$ to X_J, is a lattice as well. It is well-known that a lattice is *distributive* if and only if it does not contain the so called "pentagon lattice" or the "diamond lattice" as a sublattice.

For the sake of completeness we include a proof of the lattice property.

2.1. **Lemma.** *$(W, \preceq)$ is a lattice.*

Proof. Denote by $(w) = \{u \in W \mid u \preceq w\}$ the order ideal generated by $w \in W$. Let $w_1, w_2 \in W$. Then $(w_1) \cap (w_2)$ is the set of common prefixes of w_1 and w_2.

We claim that $(w_1) \cap (w_2) = (w)$ for some $w \in W$. If $(w_1) \cap (w_2) = \{1\}$, then $w = 1$ and we are done. Otherwise, we can choose a common prefix $s \in S$ of length 1. Then, by induction on $l(w_1)$, we have $(sw_1) \cap (sw_2) = (x)$ for some $x \in W$. Thus sx is a common prefix of w_1 and w_2. It remains to show that $p \preceq sx$ for all $p \in (w_1) \cap (w_2)$. Let $1 \neq p \in (w_1) \cap (w_2)$ and let $t \in S \cap (p)$. Again, by induction, $(tw_1) \cap (tw_2) = (y)$ for some $y \in W$ and $p \preceq ty$. Let $K = \{s, t\}$. Then the longest element w_K of W_K is a common prefix of w_1 and w_2 (see e.g., [6, (1.2.1)]). Once more by induction, we get $(w_K w_1) \cap (w_K w_2) = (z)$ for some z in W. The uniqueness of x and y requires $x = s w_K z$ and $y = t w_K z$. Therefore, $p \preceq ty = w_K z = sx$.

Let w_0 be the longest element of W and let z be the greatest lower bound of $w_0 w_1$ and $w_0 w_2$. Then $w_0 z$ is the least upper bound of w_1 and w_2. □

Observe that the existence of greatest lower bounds in W in the proof of Lemma 2.1 does not require the finiteness assumption on W.

2.2. **Lemma.** *Let $J \subseteq S$.*

(i) *The subgroup W_J is normal in W if and only if $W = W_J \times W_{S \setminus J}$.*
(ii) *If W is irreducible and $J \subsetneq S$ then the action of W on the cosets of W_J is faithful.*

Proof. (i) Let $J \subseteq S$ and suppose that W_J is normal in W. Choose $s \in J$ and $t \in S \setminus J$. Then, since $W_J^t = W_J$, we have $w = tst \in W_J$. Now tst is not a reduced expression for w, since $t \notin J$ ([6, (1.2.10)]). Hence the Exchange Condition ([6, (1.2.5)]) implies $tst = s$.

(ii) The kernel of the action of W on the cosets of W_J is the normal parabolic subgroup $H = \bigcap_{w \in W} W_J^w$ of W (cf. [6, (2.1.12)]). By (i), we must have $H = \{1\}$ since W is irreducible. □

Now suppose that W is an irreducible Weyl group, i.e., a finite crystallographic reflection group. A non-zero dominant weight λ of W is called *minuscule* provided there is no other dominant weight μ satisfying $\mu < \lambda$, where $\leq$ is the usual partial ordering on weights [5, Ch. VIII, §6.2]. The list of minuscule weights is well-known, see [5, Ch. VIII, §7.3]. For $J \subseteq S$ we call W_J *minuscule* provided W_J is the stabiliser in W of a minuscule weight. Note this definition implies that Coxeter groups of type BC_r have two distinct minuscule parabolic subgroups; one arising from the minuscule weight of the B_r root system, the other from the minuscule weight of the C_r root system.

Let Ψ be a crystallographic root system. Let Π be a set of simple roots and Ψ^+ the set of positive roots of Ψ with respect to Π. For $\beta \in \Psi^+$ we write $\beta = \sum n_\alpha(\beta)\alpha$, where the sum is taken over $\alpha \in \Pi$ and $n_\alpha(\beta) \in \mathbb{Z}_0^+$. The highest root of Ψ is denoted by ρ.

2.3. **Remark.** For every Weyl group $W = \langle S \rangle$ and for every $s \in S$ there exists a root system Ψ with Weyl group W such that $s = s_\alpha$ where α is a *long* simple root in Ψ. Throughout, given a pair W and $s \in S$, we make this choice of Ψ. As a consequence, W_J is minuscule if and only if $J = S \setminus \{s_\alpha\}$ with $n_\alpha(\rho) = 1$ in this choice of root system Ψ, by [5, Ch. VIII, §7.3].

For later reference we record the following well-known result.

2.4. **Lemma.** *Let W be an irreducible Weyl group, $\alpha \in \Pi$, and $c \in \mathbb{N}$ with $1 \leq c \leq n_\alpha(\rho)$. Then there is a unique root $\beta \in \Psi^+$ of minimal height such that $n_\alpha(\beta) = c$.*

Proof. The existence of roots $\gamma \in \Psi^+$ with $n_\alpha(\gamma) = c$ follows for instance from [4, Ch. VI, §1.6, Prop. 19]. For the uniqueness we employ an argument from the proof of [1, Lem. 1]. Let β be in Ψ^+ of minimal height such that $n_\alpha(\beta) = c$. Suppose $\beta' \in \Psi^+$ is of the same height as β and $n_\alpha(\beta) = n_\alpha(\beta')$. In particular, both β and β' are of minimal heights in their respective W_J-orbits, where $J = S \setminus \{s_\alpha\}$. Then $\langle \beta, \sigma \rangle \leq 0$ and $\langle \beta', \sigma \rangle \leq 0$ for all σ in $\Pi \setminus \{\alpha\}$. So $\{\beta\} \cup (\Pi \setminus \{\alpha\})$ is a set of pairwise obtuse positive roots and thus linearly independent. Since $n_\alpha(\beta) = n_\alpha(\beta')$, β' is in the span of $\{\beta\} \cup (\Pi \setminus \{\alpha\})$. Therefore, $\langle \beta, \beta' \rangle > 0$ (else β' could be added to give a larger independent set). Thus $\beta - \beta' = \gamma$ is a root. Without loss γ is positive. So $\beta = \beta' + \gamma$, contradicting the assumption on the heights of β and β'. ☐

Concerning further general facts and notation on Coxeter groups and root systems of Weyl groups we refer the reader to [4], [5], and [6].

3. Coset Graphs.

The coset graph Γ_J on the set of W_J-cosets of W is the directed graph with vertex set X_J and labelled edges $x \xrightarrow{s} y$ whenever $y = xs$ and $l(y) = l(x) + 1$ for $x, y \in X_J$ and $s \in S$. Thus, Γ_J is the labelled Hasse diagram of the poset $(X_J, \preceq)$. Note that $\Gamma_\varnothing$ is the Cayley graph of W with respect to the generating set S.

For $J \subseteq M \subseteq S$ denote $X_J^M = X_J \cap W_M$, the set of distinguished right coset representatives of W_J in W_M, and let $x_J^M = \sum_{x \in X_J^M} x^{-1}$. Then $X_J = X_J^M X_M$ and $x_J = x_M x_J^M$. Also, denote $a_{JKL}^M = |X_{JKL} \cap W_M|$.

3.1. **Remark.** In terms of the coset graph Γ_J, a Mackey decomposition of X_J with respect to $K \subseteq S$ ([6, (2.1.9)]) can be visualised by omitting the edges not labelled

by elements of K. This leads to the restricted coset graph

$$\Gamma_J|_K = \coprod_{d \in X_{JK}} \Gamma^K_{J^d \cap K},$$

where Γ^K_L is the graph with vertex set X^K_L (see [6, (2.2.12)]). Therefore, Γ_J contains Γ^K_L as a subgraph if $a_{JKL} > 0$.

3.2. **Remark.** If W is irreducible of rank 2, then the lattice afforded by $(W, \preceq)$ contains the pentagon lattice as a sublattice, whence it is not distributive. Therefore, by Remark 3.1, $(X_J, \preceq)$ is not distributive if $a_{JK\varnothing} > 0$ for some $K \subseteq S$ with W_K irreducible of rank 2 (i.e., if $J^d \cap K = \varnothing$ for some $d \in X_{JK}$ for such a K).

The following induction formula appears in [2].

3.3. **Lemma.** *Let* $J, K, L \subseteq S$ *and let* $K \subseteq M \subseteq S$. *Then*

$$a_{JKL} = \sum_N a_{JMN} a^M_{NKL}.$$

Proof. Using $x_K = x_M x^M_K$, we get

$$\sum_{L \subseteq S} a_{JKL} x_L = x_J x_K = x_J x_M x^M_K = \sum_{N \subseteq M} a_{JMN} x_N x^M_K = \sum_N a_{JMN} x_M x^M_N x^M_K$$
$$= \sum_N a_{JMN} x_M \sum_L a^M_{NKL} x^M_L = \sum_N \sum_L a_{JMN} a^M_{NKL} x_M x^M_L = \sum_L \Big(\sum_N a_{JMN} a^M_{NKL} \Big) x_L.$$

Hence $a_{JKL} = \sum_N a_{JMN} a^M_{NKL}$, due to the linear independence of the x_L $(L \subseteq S)$. ☐

For the remainder of the paper suppose that W is irreducible.

3.4. **Lemma.** *Let* $J, K \subseteq S$ *such that* W_K *is irreducible and* $|J| + |K| \leq |S|$. *Then* $a_{JK\varnothing} > 0$.

Proof. Assume first that $K = S$. Then $J = \varnothing$ and $a_{JK\varnothing} > 0$, since $a_{\varnothing K \varnothing} = |X_K|$.

If $K \subsetneq S$, then there exists a maximal (proper) subset M of S containing K such that W_M is irreducible, since W_K is irreducible. If $J^d \subseteq M$ for all $d \in X_{JM}$, then $W^d_J \leq W_M$ for all $d \in X_{JM}$ and so $W_J \leq W^w_M$ for all $w \in W$. But then $W_J \leq \bigcap_{w \in W} W^w_M$, the kernel of the action of W on the cosets of W_M. Now by Lemma 2.2(ii), $W_J \leq \{1\}$, i.e, $J = \varnothing$ and we are done. Suppose there is a $d \in X_{JM}$ such that $J^d \not\subseteq M$. Write $N = J^d \cap M$ (whence $a_{JMN} > 0$). Then $|N| < |J|$ and thus $|N| + |K| \leq |M|$. By induction on the rank, $a^M_{NK\varnothing} > 0$. Therefore, by Lemma 3.3, $a_{JK\varnothing} \geq a_{JMN} a^M_{NK\varnothing} > 0$. ☐

3.5. **Corollary.** *Let* $J \subseteq S$ *such that* $|J| \leq |S| - 2$. *Then* $(X_J, \preceq)$ *is not distributive.*

Proof. Clearly there exists $K \subseteq S$ such that W_K is irreducible and of rank 2. But then $a_{JK\varnothing} > 0$, by Lemma 3.4, and the result follows by Remark 3.2. ☐

Denote by $T = \{s^w \mid s \in S,\, w \in W\}$ the set of reflections in W.

3.6. **Lemma.** *Let $J \subseteq S$. Suppose there exist elements $s, t \in S$ and $r \in T$ such that $\{r, s, t\}$ generate a Coxeter group of type A_3 with $rt = tr$ and $r, s, r^s \notin W_J$. Then $(X_J, \preceq)$ is not distributive.*

Proof. Denote $K = \{s, t\} \subseteq S$. Also let d be such that $\{d\} = W_J tsr W_K \cap X_{JK}$ and let $L = J^d \cap K$. We show that $L = \varnothing$. Since W_K is of type A_2, the claim then follows from Remark 3.2.

Note that $K^{rst} = \{r, s\}$. Thus $\langle r, s\rangle = W_K^{rst}$. We write $tsr = udv$ for $u \in W_J$ and $v \in W_K$. Observe that $W_L = W_J^d \cap W_K$ (cf. [6, (2.1.12)]). Also $W_J \cap \langle r, s\rangle = \{1\}$, since it is a reflection subgroup of $\langle r, s\rangle$ contained in W_J. However, the reflections of $\langle r, s\rangle$ (which is a Coxeter group of type A_2) are $\{s, r, r^s\}$, none of which is contained in W_J by our assumption. Then $W_L^{vrst} = W_J \cap W_K^{rst} = W_J \cap \langle r, s\rangle = \{1\}$. Therefore, $W_L = \{1\}$ and thus $L = \varnothing$. □

3.7. **Lemma.** *Let W be an irreducible Weyl group of rank at least 3 and such that the highest root ρ is a fundamental weight. Let $J = \{t \in S \mid ts_\rho = s_\rho t\}$. Then $(X_J, \preceq)$ is not distributive.*

Proof. Observing [4, Planche I–IX], the choice of W implies that there is a unique simple long root σ which is not orthogonal to ρ; in particular, $J = S \setminus \{s_\sigma\}$ is maximal. Note, the condition on ρ implies that Ψ is not of type A_r or C_r and $n_\sigma(\rho) = 2$. In addition, since $|S| \geq 3$, there exists a long simple root τ which is different from and not orthogonal to σ. Set $r = s_\rho$, $s = s_\sigma$, and $t = s_\tau$. By construction, $\{r, s, t\}$ generates a reflection subgroup of W of type A_3 and $r, s \notin W_J$. It follows that $r^s = s^r \notin W_J^r = W_J$. Hence the set $\{r, s, t\}$ satisfies the conditions of Lemma 3.6 and the result follows. □

3.8. **Proposition.** *Let W be an irreducible Weyl group of rank at least 3. Let $J = S \setminus \{s_\alpha\}$ be maximal such that $n_\alpha(\rho) \geq 2$. Then $(X_J, \preceq)$ is not distributive.*

Proof. Let β be the unique root in Ψ^+ of minimal height with $n_\alpha(\beta) = 2$, cf. Lemma 2.4. Let $\Pi' = \{\delta \in \Pi \mid n_\delta(\beta) \neq 0\}$ and $\Psi' = \mathbb{Z}\Pi' \cap \Psi$. Note the Dynkin diagram of Ψ' is connected. Whence, setting $M = \{s_\delta \mid \delta \in \Pi'\} \subseteq S$, the Weyl group W_M is irreducible. Since α is long, by our choice of Ψ (cf. Remark 2.3), and $n_\alpha(\beta) = 2$, the subsystem Ψ' is not of type A_r or C_r. Thus, β is the highest root in Ψ' and α is the unique simple root in Π' not orthogonal to β. Moreover, since α is long and $n_\alpha(\beta) = 2$, W_M is of rank at least 3 (Ψ' is a subsystem of Ψ and the latter is of rank at least 3, so Ψ' is not of type G_2). Thus W_M satisfies the conditions of Lemma 3.7. In particular, there is a simple long root $\tau \in \Pi'$ which is not orthogonal to α. Setting $K = \{s_\alpha, s_\tau\}$ and $N = J \cap M$, Lemma 3.7 implies $a^M_{NK\varnothing} > 0$. Further, $a_{JMN} > 0$, as $1 \in X_{JMN}$. Now, by Lemma 3.3, we have $a_{JK\varnothing} \geq a_{JMN} a^M_{NK\varnothing} > 0$. The result follows by Remark 3.2. □

Example. We illustrate the inductive method from the proof of Proposition 3.8 by the following example. Here W is of type B_5, $\alpha = \alpha_3$, $\tau = \alpha_2$, and $\beta = \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5$. Thus $J = S \setminus \{s_{\alpha_3}\}$ and $\Pi' = \{\alpha_2, \dots, \alpha_5\}$; thus W_M is of type B_4, indicated by the dashed box. The reflection subgroup of W of type A_3 generated by s_{α_2}, s_{α_3}, and s_β is indicated by the marked nodes.

W of type B_5

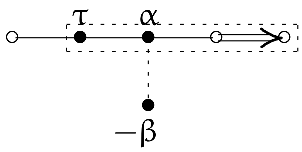


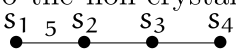
FIGURE 1. An example in type B_5

Proof of Theorem 1.1. As stated in the introduction, $(X_J, \preceq)$ is distributive for each of the cases listed in the theorem.

Now suppose the pair $W_J \leq W$ is not on this list. Thanks to Corollary 3.5, J is maximal. Since W is not dihedral, $|S| \geq 3$.

Suppose W is a Weyl group. Then W_J is not minuscule. By Remark 2.3, we have $J = S \setminus \{s_\alpha\}$, where α is a long simple root and $n_\alpha(\rho) \geq 2$. The result then follows from Proposition 3.8.

A construction similar to the one used in Lemma 3.7 can be employed to treat the non-crystallographic cases. We denote by w_J the longest element of W_J, $J \subseteq S$. In all cases we consider in Lemma 3.7, the reflection $r = s_\rho$ can be written as $r = s^{xs}$ where $s = s_\sigma$ and $x = w_N w_J$ for $N = \{u \in J \mid us = su\}$. Therefore, and since $t \in J$, we have $W_J tsr W_K = W_J sxs W_K$, for any $K \subseteq S$.

We turn to the non-crystallographic instances. Let W be of type H_4 with Dynkin diagram 

and denote $M = \{s_1, s_2, s_3\}$. Consider first the subgroup W_M of W of type H_3. We need to show for two of its three maximal parabolic subgroups W_J, that X_J^M is not distributive. We proceed as follows. Let $K = \{s_2, s_3\}$. For $s \in \{s_1, s_2\}$ let $J = M \setminus \{s\}$ and $N = \{u \in J \mid us = su\}$. Further, let $x = w_N w_J$ and $\{d\} = W_J sxs W_K \cap X_{JK}$. For $s = s_1$, we get $d = s_1 s_2 s_1$ and for $s = s_2$, we get $d = s_2 s_1$. In both cases $J^d \cap K = \varnothing$.

Now consider $W = \langle S \rangle$ of type H_4. Let $K = \{s_3, s_4\}$. For $s \in \{s_3, s_4\}$ let $J = S \setminus \{s\}$ and construct d as above: For $s = s_3$, we get $d = s_3 s_2 s_1 s_2 s_1$ and for $s = s_4$, we get $d = s_4 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2$. Again, $J^d \cap K = \varnothing$ in both cases.

Finally, if $J = \{s_2, s_3, s_4\}$ then X_J is not distributive since, by Remark 3.1, it contains the non-distributive lattice $X^M_{J \cap M}$. The same holds for $J = \{s_1, s_3, s_4\}$. $\square$

Although this is not needed for the proof of Theorem 1.1, the condition that K be irreducible in Lemma 3.4 can be dropped, as we show in Proposition 3.10 below. Corollary 3.11 suggests that this result is of independent interest.

3.9. **Lemma.** *Let* $K \subsetneq S$ *such that* $|K| \le \frac{|S|}{2}$. *Then there exist* $L \subseteq M \subsetneq S$ *such that* W_L *is conjugate to* W_K *in* W *and* W_M *is irreducible.*

Proof. The case $|S| \le 1$ is trivial. For $|S| \ge 2$ we write S as a disjoint union

$$S = \coprod_{i \in I} S_i$$

where, for $i \in I$, $S_i = \{s_{i,1}, \dots, s_{i,r_i}\}$ generates a Coxeter group of type A_{r_i} such that $(s_{i,j-1}s_{i,j})^3 = 1$ ($j = 2, \dots, r_i$), where $s_{i,j}$ commutes with $s_{i',j'}$ whenever $i \ne i'$ unless $j = j' = 1$ ($1 \le j \le r_i$, $1 \le j' \le r_{i'}$, $i, i' \in I$), where s_{i,r_i} commutes with all but one $s \in S$ ($i \in I$), and where $\langle s_{i,1} \mid i \in I \rangle$ is of type A_3 or irreducible of rank 2. (This is always possible since the Coxeter graph of W contains at most one vertex of degree more than 2 or at most one edge not of type A_2.) In this situation the maximal parabolic subgroup W_{M_i} of W, where $M_i = S \setminus \{s_{i,r_i}\}$, is irreducible ($i \in I$).

Now let $K \subseteq S$ be such that no conjugate W_L ($L \subseteq S$) of W_K is contained in an irreducible maximal parabolic subgroup of W. We have to show that then $|K| > |S|/2$.

Let $i \in I$ and suppose that $K \cap S_i$ contains neither $s_{i,j-1}$ nor $s_{i,j}$ for some $1 < j \le r_i$. Let $x = s_{i,j}s_{i,j+1} \cdots s_{i,r_i}$. Then $\{s_{i,j+1}, \dots, s_{i,r_i}\}^x = \{s_{i,j}, \dots, s_{i,r_i-1}\}$ and x leaves $K \setminus \{s_{i,j}, \dots, s_{i,r_i}\}$ fixed. Hence $K^x \subseteq S$ is a conjugate of K that does not contain s_{i,r_i} and therefore lies in M_i contradicting our choice of K. It follows for each $i \in I$ that K contains at least half of the elements of S_i, whence $|K| \ge \sum_{i \in I} |S_i|/2 = |S|/2$.

Suppose $K \cap \{s_{i,1} \mid i \in I\} = \varnothing$. By the preceding argument we then must have $r_i \ge 2$ for all $i \in I$. Let $x = s_{1,1}s_{1,2} \cdots s_{1,r_1}$. Then $\{s_{1,2}, \dots, s_{1,r_1}\}^x = \{s_{1,1}, \dots, s_{1,r_1-1}\}$ and x leaves $K \setminus S_1$ fixed. Hence $K^x \subseteq S$ is a conjugate of K that does not contain s_{1,r_1} and therefore lies in M_1 contradicting our choice of K.

It follows that $|K \cap S_i| > |S_i|/2$ for at least one $i \in I$ whence the result. □

Example. We illustrate a suitable decomposition $S = \coprod_i S_i$ from the proof of Lemma 3.9 by the following example. Here W is of type E_6. The parabolic subgroup of W of type A_3 is indicated by the marked nodes.

3.10. **Proposition.** *Let* $J, K \subseteq S$ *such that* $|J| + |K| \le |S|$. *Then* $a_{JK\varnothing} > 0$.

Proof. From $X_{KJ\varnothing} = X_{JK\varnothing}^{-1}$ it follows that $a_{JK\varnothing} = a_{KJ\varnothing}$. Thus we may assume that $|K| \le |S|/2$. Note also that $a_{JL\varnothing} = a_{JK\varnothing}$ for all conjugates $L \subseteq S$ of K (for, if $W_L = W_K^x$ for some $x \in W$, then right multiplication by x induces a bijection $W_J d W_K \mapsto W_J d W_K x = W_J dx W_L$ of double cosets and $W_J^d \cap W_K = \varnothing$ implies $W_J^{dx} \cap W_L = \varnothing$). Therefore, using Lemma 3.9, we may assume that there exists $M \subseteq S$ such that W_M is an irreducible maximal parabolic subgroup of W containing W_K. Now we proceed as in the proof of Lemma 3.4. □

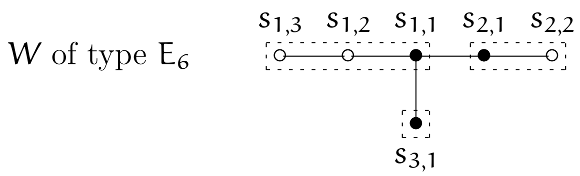


FIGURE 2. An example in type E_6

As a special case, we immediately obtain from Proposition 3.10 and Remark 3.1:

3.11. **Corollary.** *For each $s \in S$ the coset graph $\Gamma_{\{s\}}$ contains the full Cayley graph of* every *maximal parabolic subgroup of W as a subgraph.*

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